MONOTONE TRANSFORMATIONS

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In this paper we initiate the study of transformations which are monotone relative to a collection of sets. These mappings include, by proper specialization of the family of sets, both monotone and non-alternating transformations. Since we are concerned with non-metric spaces it is necessary to extend to such spaces those results of the classical theory of continua needed in our treatment. Where available proofs were adequate we have made reference to them. The first section contains general results on continua and, in particular, a proof, for non-separable spaces, of the theorem that every continuum contains at least two non-cutpoints. In the second section we give a cyclic element theory and show that if p is neither an endpoint nor a cutpoint then there is a point conjugate to p. Certain essential departures from the situation in separable spaces are noted. In the last section an extension of the result (due to Schweigert) that A-sets are invariant under non-alternating transformations is given.

1. By a topological space we mean a set S together with a class of closed sets satisfying the conditions CS of Lefschetz [7; I]. In general we adhere to the terminology of Lefschetz and Alexandroff-Hopf [1]. However, we use "compact" in the sense of "bi-compact" and in a normal space it is not necessarily assumed that a single point is a closed set.

A family of sets will be termed an *M-collection* if each set is closed and if, with each pair of sets, the collection contains their intersection. It is clear that an aggregate of closed sets is an M-collection if and only if, with each finite family of sets, it contains their intersection. A property B of closed sets will be termed *inductive* provided that, if each element of an M-collection has property B, then the intersection of all the sets has property B. Equivalently we also speak of a *collection* G as being *inductive* if each M-collection in G has its intersection in G. Manifestly a theorem about inductive collections implies a theorem about inductive properties and conversely, so that the two concepts may be used correlatively.

By a partially ordered system we mean a class P and a binary relation R between elements of P such that (i) pRp for each p in P, (ii) pRq and qRp imply $p = q$, (iii) pRq and qRr imply pRr. A subset P_0 of P will be called an ordered system if for each pair of elements p and q in P_0 we have either pRq or qRp. It is known [4; 140] that any ordered subsystem P_0 of P is contained in a *maximal* ordered subsystem of P. This result is of fundamental importance in non-separable spaces.

(1.1) *Any non-null inductive collection G of S contains a minimal element.*

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