n-TO-ONE MAPPINGS OF LINEAR GRAPHS

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1. **Introduction.** An n-to-1 continuous mapping is one for which every inverse image consists of exactly n points. Such mappings have been considered by O. G. Harrold [2], who showed that no 2-to-1 mapping can be defined on an arc. (In general, we shall use the term *mapping* to mean a continuous mapping.) J. H. Roberts [5] extended this result to a closed 2-cell and proved other theorems concerning 2-to-1 mappings defined over complete metric spaces. A paper by Roberts and Venable Martin [3] deals with such mappings of 2-dimensional manifolds. In a second paper [1] Harrold studied n-to-1 mappings on connected linear graphs.

Using the methods developed by Roberts, this paper considers first the question of defining a 2-to-1 mapping of any linear graph A. It is shown that unless the Euler characteristic $\chi(A)$ is even such a mapping cannot be defined on A. However, if $\chi(A)$ is odd, the following analogous question can be investigated. Does there exist a mapping of A which is 2-to-1 except that one inverse image consists of a single point? Γ is defined as the class of all mappings T defined over linear graphs, where T is either exactly 2-to-1 or else 2-to-1 except that one inverse image consists of a single point. In §3, it is shown that a mapping of class Γ can be defined on any linear graph which is a boundary curve and that any connected graph is the image of a boundary curve under some T belonging to Γ. In §4, the problem of the definition of n-to-1 mappings on a linear graph is considered. It is shown that if a mapping of class Γ can be defined on a linear graph A, then A admits an exactly n-to-1 mapping, for all $n \neq 2$.

2. **Two-to-one mappings.** Let T be an exactly 2-to-1 mapping defined over a linear graph A. (A linear graph is the sum of a finite number of arcs such that if a point p is common to two of the arcs, then p is an end point of each of them. Considering the end points as vertices and the arcs as 1-cells, we have a 1-dimensional complex.) The set of inverse images under T is an upper semi-continuous collection G of elements filling A, such that every element of G is a pair of points. For each point x in A, let $s(x)$ be the other point in the element. For any subset M of A, let $s(M)$ be the set of all points $s(x)$ for which x is in M. Let $f(x) = \rho(x, s(x))$, where ρ is the metric in A. Let K be the subset of A consisting of the points at which f is continuous. It follows from the upper semi-continuity of G that as x approaches a point q along an arc in K, $f(x)$ approaches

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