NON-ANALYTIC CLASS FIELD THEORY AND GRUNWALD'S THEOREM

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Introduction. I present here a new, simple, and non-analytic proof of the theorem of Grunwald [6] on the existence of cyclic fields of minimal degree with given local properties—the theorem which is applied in the proof that every division algebra is cyclic. I also include a systematic exposition of the main part of class field theory, from the index theorems on, in the considerably simplified form made possible by proving the theorems directly in terms of the ideal elements, and the Artin-symbol for ideal elements, introduced by Chevalley [1]. No such account exists in the literature, since the non-analytic proofs recently given by Chevalley [2] are expressed in terms of topology, infinite Abelian extensions, and group characters. This exposition includes non-analytic proofs of the theorem that every class field is Abelian and of the norm index theorem for general fields, a very simple proof of the theorem on possible values of the norm residue symbol, and a proof of the theorem on ramification of class fields (which is omitted in [2]). The proof of this ramification theorem is new, and somewhat shorter than that which would be obtained by the method of Herbrand and Chevalley [3].

1. **Definitions.** Let k be any finite algebraic extension of the rational field. A prime divisor p of k is a symbol associated with a valuation $|\ |_p$ of k. (For the theory of valuations, see [8], [11; X], or [12; III].) The field obtained by adjoining to k all limits of Cauchy sequences (with respect to $|\ |_p$) will be called the p-adic completion of k and will be denoted by the symbol k_p. An ideal element of k (k-idèle) is a vector $\mathfrak{a} = (\alpha_p)$, where p runs through all prime divisors of k, each α_p is a non-zero element of k_p, and α_p is a p-adic unit for all but a finite number of p. Among all prime divisors are, of course, included the infinite prime divisors (those associated with Archimedean valuations). The number α_p is called the p-component of $\mathfrak{a}$. The product of two ideal elements is formed by taking products of their p-components; it is again an ideal element. If K is a finite algebraic extension of k and $\mathfrak{A} = (\mathrm{A}_P)$ is an ideal element of K (P runs through all prime divisors of K), then $N_{K|k}\mathfrak{A}$ is by definition the ideal element of k with p-components

$$\alpha_p = \prod_{P|p} N_{K_P|k_p} \mathrm{A}_P .$$

If σ is an automorphism of K, we define the valuation $|\ |_{P^\sigma}$ of K by the equation $|\mathrm{A}^\sigma|_{P^\sigma} = |\mathrm{A}|_P$. σ can be extended to an isomorphism of K_P to K_{P^σ} as follows: if an element A_P of K_P is the limit of a sequence A_ν of elements of K, which con-

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