THE DISTRIBUTION OF PRIMES

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1. **Simple prime factors.** If $f(n)$ is a function of the positive integer n, let f_t denote the set of the solutions n of $f(n) = t$, and let $f_t(x)$ be the number of those elements of f_t which are less than x.

Thus, if $f(n)$ is the number of distinct primes dividing n or is 0 according as n is or is not square-free, then n is in f_0 if and only if it is not square-free so that $f_0(x) \sim (1 - \zeta(2)^{-1})x$ as $x \to \infty$. On the other hand, if $m > 0$, then $f_m(x)$ is the number of those integers n less than x which are composed of exactly m distinct prime factors, a number usually denoted by $\pi_m(x)$. Apparently, it was observed already by Gauss[1] that the prime number theorem, i.e., $\pi_1(x) \sim x(\log x)^{-1}$, implies, for every fixed m ($= 1, 2, \cdots$), the asymptotic relation

$$\pi_m(x) \sim L_m(x), \tag{1}$$

where

$$L_m(x) = \frac{x(\log x)^{-1}(\log \log x)^{m-1}}{(m-1)!}.$$

Thus $L_1(x) + L_2(x) + \cdots \equiv x$, although

$$\pi_1(x) + \pi_2(x) + \cdots \equiv [x] - f_0(x) \sim x/\zeta(2).$$

The latter anomaly presents itself also in case of the function $f(n) = \theta(n)$ which plays a central rôle in the following considerations and represents the number of *simple* prime factors of n (for instance, $\theta(15) = 2$, $\theta(60) = 2$, $\theta(24) = 1$). Clearly, there exists for every n exactly one m for which the set θ_m contains n so that $\theta_1(x) + \theta_2(x) + \cdots \equiv [x] \sim x$. However, for every fixed m,

$$\theta_m(x) \sim \text{const. } L_m(x), \tag{2}$$

where

$$\text{const.} = \frac{\zeta(2)\zeta(3)}{\zeta(6)}.$$

In fact, if m is fixed, an n is in θ_m if and only if $n = p_1 \cdots p_m j$ holds for m distinct primes $p_1, \cdots, p_m$ and for a j having only multiple prime factors each of which is distinct from $p_1, \cdots, p_m$. Since $\pi_m(x)$ is the number of those integers less than x which are of the form $p_1 \cdots p_m$, it follows that, in order to pass from (1) to (2), it is sufficient to show that $\sum 1/i$ has a finite

Received January 7, 1942.

[1] C. F. Gauss, *Werke*, vol. 10, part 1, 1917, p. 11 and p. 17. For the remainder term, cf. E. Landau, *Über die Verteilung der Zahlen, welche aus ν Primfaktoren zusammengesetzt sind*, Göttingen Nachrichten, 1911, pp. 361–381.