GENERALIZED "SANDWICH" THEOREMS

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The following theorem is well known under the self-explanatory name of the "ham sandwich theorem".[1]

Given any three sets in space, each of finite outer Lebesgue measure (m^*), *there exists a plane which bisects all three sets*, in the sense that the part of each set which lies on one side of the plane has the same outer measure as the part of the same set which lies on the other side of the plane.

The usual proof is based on the following theorem of Borsuk.[2]

If ϕ is a continuous mapping of the n-sphere S^n in Euclidean n-space R^n which is "antipodal" (i.e., diametrically opposite points of S^n map into points symmetric about the origin in R^n), then there is a point of S^n which maps into the origin of R^n.

If now p denotes a plane in R^3, let p^+ and p^- denote the two parts into which p divides R^3, and let v be the unit-vector perpendicular to p, oriented from p^- to p^+. Let A_i ($i = 1, 2, 3$) be the given sets. The usual argument proves first, from measure-theoretic considerations, that for each v a corresponding p can be found, depending continuously on v, which bisects A_3. The correspondence $\phi(v) = [m^*(p^+ \cdot A_1) - m^*(p^- \cdot A_1), m^*(p^+ \cdot A_2) - m^*(p^- \cdot A_2)]$ is then an antipodal mapping of S^2 in R^2. The result now follows from the case $n = 2$ of the Borsuk theorem (which can, for $n = 2$, be proved readily ab initio).

Now, a fuller use of the Borsuk theorem gives an easier proof of a more general theorem. Let R be any point-set on which a Carathéodory outer measure m^* is defined. Let f be a real-valued function defined over $S^n \times R$ such that:

(1) For each $\Lambda \epsilon S^n$, $f(\Lambda, x)$ is a measurable function over R ($x \epsilon R$), and vanishes only over a set of measure zero.

(2) For each $x \epsilon R$, $f(\Lambda, x)$ is a continuous function over S^n.

(3) For each pair of diametrically opposite points Λ and $-\Lambda$ of S^n, $f(\Lambda, x) \cdot f(-\Lambda, x) \leqq 0$ almost everywhere in R.

Write $f^+(\Lambda)$, $f^0(\Lambda)$, and $f^-(\Lambda)$ respectively for the subsets of R on which $f(\Lambda, x) >0, = 0$, and < 0. We say "*$f^0(\Lambda)$ bisects $A \subset R$*" if $m^*(f^+(\Lambda) \cdot A) = m^*(f^-(\Lambda) \cdot A)$.

Theorem. *Given any n sets $A_1, A_2, \cdots, A_n$ in R, each of finite outer measure, there exists $\Lambda \epsilon S^n$ such that $f^0(\Lambda)$ bisects each A_i ($i = 1, 2, \cdots, n$).*

Proof. Define a mapping ϕ of S^n in R^n by:

(4) The i-th coordinate of $\phi(\Lambda)$ is $m^*(f^+(\Lambda) \cdot A_i) - m^*(f^-(\Lambda) \cdot A_i)$.

Received November 12, 1941.

[1] Discovered by S. Ulam; we are indebted to the referee for calling this fact to our attention.

[2] Equivalent to Satz II of *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fundamenta Mathematicae, vol. 20(1933), p. 177. This theorem was suggested by Ulam.