MAXIMAL FIELDS WITH VALUATIONS

By Irving Kaplansky

1. **Introduction.** A field with a valuation is said to be *maximal* if it possesses no proper immediate extensions, i.e., if every extension of the field must enlarge either the value group or the residue class field. This definition is due to F. K. Schmidt, but was first published by Krull ([4], p. 191). In the same paper Krull succeeded in proving that any field with a valuation possesses at least one immediate maximal extension and that any field of formal power series is maximal in its natural valuation. These facts led Krull to propound the following two queries.

(1) Is the immediate maximal extension of a field uniquely determined?

(2) If a maximal field K has the same characteristic as its residue class field, is K necessarily a power series field?

These two closely related questions form the central problem of this investigation. The answer to the first is obtained in §3 (Theorem 5), as follows. The immediate maximal extension is always unique if the residue class field has characteristic ∞; but if the latter has characteristic p, then a pair of conditions which we have labelled "hypothesis A" must be satisfied. It is then not difficult to obtain the answer to the second question in §4. In fact, with the same hypothesis, the answer is again affirmative, provided factor sets are admitted in the construction of the power series field (Theorem 6). Granted an additional hypothesis, it is furthermore possible to dispense with factor sets (Theorem 8). In §5, examples are given to show that the conclusions of the preceding theorems may fail if hypothesis A is not fulfilled.

The notion of pseudo-convergence, borrowed from Ostrowski ([9], p. 368), appears to be a natural tool for investigations of maximality, and it is employed consistently throughout the paper. The reason for this is to be found in Theorem 4, which shows that pseudo-convergence provides us with an *intrinsic* characterization of maximality.

2. **Pseudo-convergence and maximality.** Throughout this section K will always denote a field with a valuation V on an ordered Abelian group Γ, B its valuation ring, and $\mathfrak{K}$ its residue class field.[1]

Definition. A well-ordered set $\{a_\rho\}$ of elements of K, without a last element, is said to be *pseudo-convergent* if

(1) $$V(a_\sigma - a_\rho) < V(a_\tau - a_\sigma)$$

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[1] For these definitions, cf. [4] and [7].