GENERALIZED ARITHMETIC

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1. **Introduction.** Since the time of Cantor, it has been the fashion to divide all arithmetic at the root into two separate branches: cardinal and ordinal. Each of these branches is supposed to have its peculiar operations of addition, multiplication, and exponentiation. Only as an afterthought are the two branches connected, by a roughly[1] homomorphic correspondence from ordinal arithmetic to cardinal arithmetic, which is ismorphic when restricted to finite ordinals and cardinals.

In the present paper, an entirely different point of view is advanced. Instead of giving finite and transfinite arithmetic a split personality, half ordinal, half cardinal, I believe that one should regard both aspects as fragments of a unified general arithmetic of partially ordered systems.

I should like to stress three arguments in favor of this point of view.

In the first place, what are usually considered as purely cardinal operations extend in a natural way to ordinal numbers and other partially ordered systems, and vice versa. Moreover, when applied to the wider context of general partially ordered systems, the six operations of "generalized arithmetic" are found to have important new applications.[2] The variety and importance of these will stand comparison with the applications of transfinite arithmetic, as that term has been understood heretofore.[3]

In the second place, almost all arithmetical laws which are valid in transfinite arithmetic, as that term is understood now, are equally valid when the operations are applied to the most general partially ordered systems. In fact, the big gap comes between ordinary arithmetic and transfinite arithmetic; much more is lost by admitting infinite numbers as legitimate objects for arithmetic operations than is lost by including partially ordered sets in the middle ground between totally ordered sets (finite ordinals) and totally unordered sets (finite cardinals). Moreover, the slight loss is more than compensated by the availability of *new cross-laws* connecting cardinal with ordinal operations.

Finally, adoption of the broader point of view towards arithmetic developed b low fits the traditional transfinite arithmetic into the general framework of

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[1] Ordinal exponentiation does not quite fit into this statement, and involves special complications. Also, the proof of this connection involves the well-ordering principle (axiom of choice).

[2] Much may be found about the extension of cardinal operations and applications of the extended definitions in [1]. However, the scope of the present program is nowhere suggested in that paper.

[3] The need for the *operations* of transfinite arithmetic was never very great; the need in topology for even transfinite ordinals has now largely disappeared, thanks to the increased use of the more effective and simpler tool of Moore-Smith convergence.