A SELF-RECIPROCAL FUNCTION

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The object of this paper is to establish the following theorem.

The function

$$f(x) = x^{\nu-p-q+\frac{1}{2}}e^{-\frac{1}{4}x^2}J_p(x)J_q(x)$$

is[1] R_ν, *provided* $\Re(\nu) > -1$.

We shall require the integral

$$I = \int_0^\infty x^{s-1}e^{-\frac{1}{4}x^2}W_{k,m}(\tfrac{1}{2}x^2)J_p(ax)J_q(ax)\,dx.$$

This can be evaluated by substituting for $J_p(ax)J_q(ax)$ the equivalent infinite series (see [4], p. 380)

$$\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(p+q+2r+1)(\frac{1}{2}ax)^{p+q+2r}}{r!\,\Gamma(p+r+1)\Gamma(q+r+1)\Gamma(p+q+r+1)}$$

and integrating term by term by the help of the integral (see [2])

$$\int_0^\infty x^{l-1}e^{-(\alpha^2+\frac{1}{2})x}W_{k,m}(x)\,dx$$

$$= \frac{\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}\,{}_2F_1(l+m+\tfrac{1}{2}, l-m+\tfrac{1}{2}; l-k+1; -\alpha^2)$$

$$(l \pm m + \tfrac{1}{2} > 0, \ |\Im(\alpha)| < 1).$$

We then obtain

$$I = (\tfrac{1}{2}a)^{p+q}\,\frac{2^{\frac{1}{2}s-1}\Gamma(\frac{1}{2}s+\frac{1}{2}p+\frac{1}{2}q+m+\frac{1}{2})\Gamma(\frac{1}{2}s+\frac{1}{2}p+\frac{1}{2}q-m+\frac{1}{2})}{\Gamma(p+1)\Gamma(q+1)\Gamma(\frac{1}{2}s+\frac{1}{2}p+\frac{1}{2}q-k+1)}$$

$$(1) \qquad \times\, {}_4F_4\left[\begin{matrix}\frac{1}{2}p+\frac{1}{2}q+\frac{1}{2}, & \frac{1}{2}p+\frac{1}{2}q+1, & \frac{1}{2}s+\frac{1}{2}p+\frac{1}{2}q+m+\frac{1}{2}, & \frac{1}{2}s+\frac{1}{2}p+\frac{1}{2}q-m+\frac{1}{2}; & -2a^2\\ p+1, & q+1, & p+q+1, & \frac{1}{2}s+\frac{1}{2}p+\frac{1}{2}q-k+1 & \end{matrix}\right].$$

Term by term integration is justified by virtue of

$$(2) \qquad |W_{k,m}(x)| = O(e^{-\frac{1}{2}x}x^k), \qquad |J_\nu(x)| = O(x^{-\frac{1}{2}})$$

and by virtue of the size of the terms in the series of $J_p(ax)J_q(ax)$. Hence the result (1) has been shown to be true when $\Re(p) > -1$, $\Re(q) > -1$, and

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[1] Following Hardy and Littlewood, we say that a function is R_ν when it is self-reciprocal in the Hankel-transform of order ν.