ASSOCIATED DOUBLE INTEGRAL VARIATION PROBLEMS

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Introduction

In a paper entitled *Über adjungierte Variationsprobleme und adjungierte Extremalflächen*, Haar [1][1] has given a variation problem associated with a non-parametric double integral variation problem of the type

$$(1) \qquad J[z] = \iint_R F(p, q)\, dx\, dy = \text{min.}, \qquad p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y},$$

in such a way that an extremal surface of the problem (1) determines an extremal surface of the associated problem; and a variation problem associated with a parametric double integral variation problem of the type

$$(2) \qquad I[x, y, z] = \iint_G \Phi(A, B, C)\, du\, dv = \text{min.},$$
$$A = \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}, \qquad B = \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix}, \qquad C = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix},$$

in such a way that an extremal surface of the problem (2) determines an extremal surface of the associated problem. It is the purpose of this paper to show that the method used by Haar to determine such associated problems can be used to determine a group of such associated problems. With this end in view we give a summary of the results of Haar.

0.1. **Non-parametric adjoint variation problem of Haar.** Let us assume that the integrand function $F(p, q)$ of the problem (1) is of class[2] C'' in a region S of the pq-plane and define the functions

$$(3) \qquad \begin{aligned} X(p, q) &= -F_p(p, q), & Y(p, q) &= -F_q(p, q), \\ Z(p, q) &= F - pF_p - qF_q, & \Delta(p, q) &= F_{pp}F_{qq} - F_{pq}^2 . \end{aligned}$$

Let us further assume that the functions $F(p, q)$, $Z(p, q)$, $\Delta(p, q)$ are different from zero everywhere in S and that the transformation[3]

$$T_3\colon \quad p_3 = -\frac{X(p, q)}{Z(p, q)}, \qquad q_3 = -\frac{Y(p, q)}{Z(p, q)}, \qquad F_3(p_3, q_3) = -\frac{1}{Z(p, q)},$$

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[1] Numbers in square brackets refer to the bibliography given at the end of this paper.

[2] A function is said to be of class $C^{(n)}$ in a region if the function together with its partial derivatives up to and including those of the n-th order are continuous in the region. A surface $z = z(x,y)$ is said to be of class $C^{(n)}$ in a region of the xy-plane if $z(x,y)$ is of class $C^{(n)}$ in the region.

[3] The particular choice of subscripts here used is for reference in later work.