A GENERAL KUMMER THEORY FOR FUNCTION FIELDS

BY SAUNDERS MAC LANE AND O. F. G. SCHILLING

INTRODUCTION

Kummer theory studies the generation of Abelian fields by radicals over a given base field F. This paper will develop a "relative" form of the theory, in which the base field F is a field of algebraic functions over a coefficient field. The modified theory then considers extensions of this F which are generated by radicals, or by arbitrary algebraic extensions of the coefficient field, or both. The normal extensions of this type, unlike the ordinary Kummer extensions, are in general non-Abelian.

This development was suggested by an attempt to generalize the principal ideal theorem of algebraic number theory. Schmidt, Hasse, and others [17], [8][1] have shown that the class field theory over algebraic number fields has a strict analogue for function fields of one variable over a finite coefficient field. One might then surmise that the principal ideal theorem of Hilbert [2], [7] has a similar analogue. We may phrase this conjecture as follows. Given the group of divisor classes of degree zero in F, does there exist an unramified Abelian extension K of F whose Galois group is isomorphic to the group of divisor classes and in which all these divisors become principal? We shall show that this is not the case.

First, we describe more explicitly the behavior of divisor classes which become principal in an extension, and show that the behavior of such principal divisors can be restated in an elementary fashion, free of arithmetic concepts.

Consider an algebraic function field F of one variable, over a field $\mathfrak{F}$ of constants (in the classical case, $\mathfrak{F}$ = the complex numbers). The field F has an abstract Riemann surface whose points P can be described as prime divisors; i.e., as homomorphic mappings of F on $\mathfrak{F}'$ plus ∞, where $\mathfrak{F}'$ is an algebraic extension of the field $\mathfrak{F}$ of constants. Each function x of F has a finite number of zeros and poles at various prime divisors P_i. With the proper multiplicities (positive for zeros, negative for poles) these may be listed as a formal product:

$$(x) = P_1^{e_1}P_2^{e_2} \cdots P_n^{e_n}.$$

This product is the *divisor* of x. Any such formal product $A = \prod P_i^{e_i}$ is called a divisor, though it need not be the divisor of any function x of the field. The sum $\sum e_i f_i$ is the *degree* of A, if $f_i = [\mathfrak{F}_i:\mathfrak{F}_i]$. The divisors of the form $A = (x)$ are called *principal* divisors; they always have degree zero (number of zeros = number of poles). If K is any finite algebraic extension of the given function field F, each divisor of F may be construed as a suitable divisor of K,

Received October 24, 1941; presented to the American Mathematical Society on May 2, 1941 under the title "The principal divisor theorem for function fields".

[1] Numbers in square brackets refer to the bibliography at the end of the paper.