POSITIVE DEFINITE FUNCTIONS ON SPHERES

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1. **Introduction.** Let S_2 denote the ordinary spherical shell of radius one and center o and let $p_1, p_2, \cdots, p_n$ be n arbitrary points of S_2. Let p_ip_k denote the spherical distance between the points p_i, p_k. For n real variables $x_1, x_2, \cdots, x_n$ we have the following inequality

$$(1.1) \qquad |\sum \overrightarrow{op_i} \cdot x_i|^2 = \sum_{i,k=1}^{n} \cos (p_i p_k) x_i x_k \geqq 0,$$

which is equivalent to the determinant inequality

$$\det | \cos (p_i p_k) |_{1,n} \geqq 0$$

for arbitrary points p_i and arbitrary n. This property of the function $g(t) = \cos t$ in relation to the space S_2 is expressed by saying that $\cos t$ is positive definite in S_2. The general definition is as follows. Let M be a metric space with the distance function pq. A real continuous function $g(t)$ $(0 \leqq t \leqq$ diameter of $M)$ is said to be positive definite (p. d.) in M if we have

$$(1.2) \qquad \sum_{i,k=1}^{n} g(p_i p_k) x_i x_k \geqq 0,$$

for any n points $p_1, \cdots, p_n$ of M, arbitrary real x_i, and all $n = 2, 3, \cdots$.

We denote this class of functions by the symbol $\mathfrak{P}(M)$. It enjoys the following useful closure properties:[1]

I. If $g_1(t) \in \mathfrak{P}(M)$, $g_2(t) \in \mathfrak{P}(M)$, also $c_1 g_1(t) + c_2 g_2(t) \in \mathfrak{P}(M)$, provided $c_1 \geqq 0$, $c_2 \geqq 0$.

II. The same assumptions imply also that $g_1(t) g_2(t) \in \mathfrak{P}(M)$.

III. If $g_n(t) \in \mathfrak{P}(M)$, $g_n(t) \to g(t)$ as $n \to \infty$, and $g(t)$ is continuous, then also $g(t) \in \mathfrak{P}(M)$.

In the present note we are concerned with the classes $\mathfrak{P}(S_m)$ and $\mathfrak{P}(S_\infty)$ corresponding to the unit spheres in the Euclidean space E_{m+1} and the Hilbert space H respectively.

Returning to S_2, we have noticed that $\cos t \in \mathfrak{P}(S_2)$. It will be shown below that also $P_n(\cos t) \in \mathfrak{P}(S_2)$, where P_n is a Legendre polynomial. By the above mentioned closure properties it is now apparent that also

$$(1.3) \qquad g(t) = \sum_{n=0}^{\infty} a_n P_n(\cos t) \in \mathfrak{P}(S_2),$$

provided $a_n \geqq 0$ $(n = 0, 1, \cdots)$ and $\sum a_n$ converges. This formula will be shown to furnish the most general element of $\mathfrak{P}(S_2)$.

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[1] See [6], §2. Numbers in brackets refer to the bibliography at the end of the paper.