A GENERALIZATION OF THE EUCLIDEAN ALGORITHM TO SEVERAL DIMENSIONS

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Summary. The Euclidean algorithm is generalized to two, three, and four dimensions. The generalized algorithm is applied to the solution of the following problems.

Given a positive definite quadratic form, find integer values of the variables, not all zero, which make the value of the form a minimum.

Given n linear forms in n variables with determinant $\Delta \neq 0$, find integer values of the variables, not all zero, such that each linear form $\leqq |\Delta|^{1/n}$ in absolute value.

Given n real numbers, $x_1, \cdots, x_n$, not all rational, find as many sets, $a, a_1, a_2, \cdots, a_n$, of integers as desired such that simultaneously

$$| ax_i - a_i | \leqq a^{-1/n} \qquad (i = 1, 2, \cdots, n).$$

Given

$$L = \sum_{i=1}^{n} a_i x_i,$$

the a_i's being coprime integers, find a general solution, in integers, of $L = k_1$, namely

$$x_i = \sum_{j=1}^{n} b_{ij} k_j \qquad (i = 1, \cdots, n)$$

(where the b's are fixed integers, k_1 is the same integer that occurs in $L = k_1$, and the other k's are arbitrary integers) such that $\sum (b_{ij})^2$ shall be a minimum.

Given a hypersphere, σ, with center at the origin and radius $\geqq 1$, and

$$L = \sum_{i=1}^{n} u_i x_i,$$

the u's being real numbers, find a lattice point distinct from the origin within (or on) σ and as close to the hyperplane $L = 0$ as possible.

Given two symmetric positive definite matrices, A and B, with real components, find whether there is a matrix P with integral components and determinant ± 1 such that $B = P^T AP$, and if so, to find all such P's.

We open the paper with some preliminary conventions regarding terminology.

We shall use lower case italics from a to t inclusive for rational integers, and from u to z inclusive for real numbers. We shall use upper case italics from A to Q inclusive for square matrices with real elements, and from R to Z in-

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