THE FUCHSIAN EQUATION OF SECOND ORDER WITH FOUR SINGULARITIES

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1. In this paper the solutions of the Fuchsian equation of second order with four singularities are investigated by means of series of hypergeometric functions.

A linear differential equation of second order with four singularities which are "regular points" (this name is due to Thomé whereas Fuchs himself used the term "points of determinateness"; both names appear to be rather inadequate) can be reduced by a linear transformation of the variables to the equation defined by the Riemannian scheme

$$P\begin{Bmatrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & \alpha & x \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{Bmatrix}, \tag{1.1}$$

where the exponents are connected by Riemann's relation

$$\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0. \tag{1.2}$$

Heun's equation defined by the scheme (1.1) is of considerable theoretical interest, for it is the simplest equation of Fuchsian type the coefficients of which are not determined uniquely by the singularities and the exponents attached to the singularities. In fact, in Heun's equation (3.1) there is a constant h which is quite arbitrary from the point of view of the scheme (1.1) and is thus an accessory parameter according to the terminology of F. Klein. From the practical point of view, Heun's equation is of some interest, for many of the differential equations occurring in the applications of analysis are special or limiting cases of Heun's equation. It is sufficient to recall that the hypergeometric and confluent hypergeometric equations, the differential equations of Lamé, Mathieu, Legendre, Bessel and Weber, those of the polynomials of Jacobi, Tchebicheff, Laguerre and Hermite as well as that of Bateman's k-function belong to this class.

2. The simplest way of representing the fundamental branches of the functions defined by the scheme (1.1) is to try power-series. These represent the functions in a circle with one singularity on the boundary and another singularity at the center of the domain of convergence. An alternative plan, proposed in this paper, consists in expanding the solutions of Heun's equation into certain series of hypergeometric functions. In this way, in a certain sense, three singularities may be taken into consideration. The series are convergent in a domain the boundary of which is an elliptic limaçon with two singularities as foci and a third singularity on the circumference. Though these series may offer some points of interest even in the general case (i.e., with arbitrary values

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