CLASSIFICATION OF SOLUTIONS AND OF PAIRS OF SOLUTIONS OF $y''' + 2py' + p'y = 0$ BY MEANS OF INITIAL CONDITIONS

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The following facts concerning real solutions of the real differential equation $y''' + 2py' + p'y = 0$ are either explicitly stated in or are readily deducible from the work of G. D. Birkhoff [Annals of Mathematics, (2), vol. 12(1911), pp. 103–127]. There are non-vanishing solutions. If a solution has a double zero,[1] it has no simple zeros. If each of two linearly independent solutions has double zeros, their zeros interlace along the x-axis. If one of two solutions has double zeros while the second has simple zeros, either each zero of the first coincides with a zero of the second, there being exactly one zero of the second between successive coincidences, or the solutions have no zeros in common, there being exactly two zeros of the second between successive zeros of the first. If each of two linearly independent solutions has simple zeros, either their zeros interlace, or they do so in pairs, or alternate zeros of the first coincide with alternate zeros of the second.

It is the purpose of this paper to distinguish between the three types of solutions, and between the various situations with regard to two solutions, by means of numbers determined by the values of the solutions and their first two derivatives at any given point. In order to achieve this end, it is necessary to demonstrate in a manner different from that of Birkhoff that the above statements are true.

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Let $p \equiv p(x)$, on the interval (a, b), be a real continuous function of the real variable x, with a continuous derivative. Consider

$$y''' + 2py' + p'y = 0. \tag{1}$$

We postulate that some solution of this equation has at least two double zeros on (a, b). It will become apparent in the course of discussion that this is equivalent to demanding that some solution have at least three zeros on (a, b)

Let y_i and y_j be any two solutions of (1). Then

$$y_i(y_j''' + 2py_j' + p'y_j) + y_j(y_i''' + 2py_i' + p'y_i)$$
$$= y_iy_j''' + y_i'''y_j + 2py_iy_j' + 2py_i'y_j + 2p'y_iy_j = 0.$$

By integration we obtain

$$y_iy_j'' + y_i''y_j - y_i'y_j' + 2py_iy_j = C_{ij}, \tag{2}$$

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[1] The terms "double zero" and "simple zero" are used in the sense that x_1 is a double zero of y_1 if $y_1(x_1) = y_1'(x_1) = 0$, $y_1''(x_1) \neq 0$, and x_1 is a simple zero of y_2 if $y_2(x_1) = 0$, $y_2'(x_1) \neq 0$.