A COMPARISON OF LINEAR MEASURES IN THE PLANE

By Seymour Sherman

In generalizing from the notion of the length of curve and linear measure of a linear set to the linear measure of a plane set the following considerations arise:

1. Does the new measure give the expected results for point sets which can be treated by the old methods?

2. Is the new measure invariant under Euclidean transformation of the set?

3. Does the new measure have the usual measure properties, i.e., is it completely additive; does it satisfy the general Carathéodory measure postulates?

Some such generalizations satisfying these and more subtle[1] requirements have been proposed by Carathéodory, Gross, Steinhaus,[2] Favard, Kolmogoroff,[3] Appert, Randolph, and Morse. Of these measures the ones associated with Carathéodory, Gross, Appert, Randolph, and Morse are closely related and involve countable decompositions of the given set while the ones suggested by Kolmogoroff and Steinhaus diverged into different paths. Kolmogoroff measure originated, in part, with the notion of Schmidt [10] that the measure of a contracted set is not greater than the measure of the set. Steinhaus measure originated (1) with the measure (see footnote 6) of sets of lines used for the Buffon needle problem and (2), surprisingly enough, with a mechanical device[4] for measuring lengths of curves as seen under a microscope.

We are mainly concerned with the relationship between Carathéodory linear measure and Steinhaus linear measure. We complete Steinhaus' proof that the Steinhaus measure of a rectifiable Jordan curve is equal to twice its length as defined by the inscribed polygon approach. In Theorem 4, we show that, contrary to Steinhaus' expressed belief,[5] there are sets (irregular in the sense of Besicovitch) whose Steinhaus measure is different from twice their Carathéodory linear measure, and, in general, if a set is measurable Carathéodory, then its Steinhaus linear measure is equal to twice the Carathéodory linear measure of its regular part. Unlike other linear measures, Steinhaus linear measure may

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[1] One such requirement is that the outer linear Lebesgue measure of the projection of the set on any line be not greater than the outer measure of the set. For a discussion of this requirement see [5]. Numbers in square brackets refer to the bibliography. For considerations involving a natural generalization to higher dimensions see [7] and [8].

[2] See [12] and [13]. This measure is completely additive over the family of sets measurable Steinhaus but, since it is not introduced by means of an exterior measure function, the Carathéodory measure postulates do not apply. This measure was later independently suggested by Favard [4].

[3] Kolmogoroff measure [6] is defined merely for analytic sets and so the general Carathéodory postulates do not apply. It is completely additive over analytic sets.

[4] See [12].

[5] See [13], p. 354.