# A PROPERTY OF BANACH SPACES

By Mahlon M. Day

By a method involving a general form of integration, due to Hildebrandt, Goldstine [3] proved that a certain kind of weak completeness is necessary and sufficient for reflexivity of a Banach space $B$.[1] The present note gives for this theorem a new proof, based on a simple geometrical property (Lemma 1) of every Banach space. The nature of the proof suggests a new criterion (Theorem 2) for reflexivity of a Banach space; Theorem 3 collects a number of criteria for reflexivity of every $\lambda$-separable subspace of a Banach space.

If $E$ is a subset of $B$, let $r(E) = \inf_{b \in E} \| b \|$ with the usual convention that if $E$ is empty, $r(E) = +\infty$.

Lemma 1. *If $\beta_1, \cdots, \beta_k \in B^*$, if $E = \{b \mid \beta_i(b) = c_i \text{ for } i = 1, \cdots, k\}$ and if $M = \sup | \Sigma_{i \leqq k} t_i c_i |/\| \Sigma_{i<k} t_i \beta_i \|$ where the supremum is taken*[2] *over all choices of the real numbers $t_1, \cdots, t_k$, then $M = r(E)$.*

This is essentially Helly's theorem; a short proof due to Mimura is quoted in Kakutani [4].

A set $X$ of elements $x$ is called *directed* by a relation $>$ if (1) $x_1 > x_2$ and $x_2 > x_3$ imply that $x_1 > x_3$, and (2) if $x_1$ and $x_2$ are in $X$ there is an $x_3$ in $X$ such that $x_3 > x_1$ and $x_3 > x_2$.[3] If $Y$ is any topological space and $f$ any function on $X$ to $Y$, $f$ converges to $y$ or $y = \lim_x f(x)$ if and only if for each neighborhood $N$ of $y$ there is an $x_N$ in $X$ such that $f(x) \in N$ if $x > x_N$. Following Goldstine, say that $B$ is *weakly complete relative to* $X$ if the conditions (1) $\| b_x \| \leqq K$ for every $X$, and (2) $\lim_x \beta(b_x)$ exists for every $\beta$ in $B^*$, together imply that a $b$ in $B$ exists for which $\lim_x \beta(b_x) = \beta(b)$ for every $\beta$ in $B^*$. $B$ is *weakly complete* if it is weakly complete relative to every directed set $X$.

The *weak neighborhoods* of a point $b_0$ in $B$ are the sets

$$N = N(b_0 ; \beta_1, \cdots, \beta_k ; \epsilon) = \{b \mid | \beta_i(b) - \beta_i(b_0) | < \epsilon \text{ for } i = 1, \cdots, k\}$$

for every choice of $\epsilon > 0$, the integer $k$, and the points $\beta_1, \cdots, \beta_k$ in $B^*$. The weak topology can be defined in the same way in a conjugate space $B^*$ but another topology is often more useful. The *weak* neighborhoods* of $\beta_0$ in $B^*$ are

Received February 3, 1941.

[1] It is assumed that the reader is familiar with the definition of a Banach space $B$, and its conjugate space $B^*$; see for example Banach [1]. $B$ is reflexive if for each $\mathfrak{b}$ in $B^{**}$, the second conjugate space of $B$, there is a $b$ in $B$ such that $\mathfrak{b}(\beta) = \beta(b)$ for all $\beta$ in $B^*$.

[2] $\{x \mid \cdots\}$ means the set of all $x$ satisfying the conditions following the vertical bar. For $a_1, a_2 \geqq 0$ we shall make the convention that no matter what $a_2$ is, $a_1/a_2 = 0$ if $a_1 = 0$; also $a_1/a_2 = +\infty$ if $a_2 = 0$ and $a_1 > 0$.

[3] Directed sets were first studied by Moore and Smith [5]; G. Birkhoff [2] adapted this notion of convergence to topological uses.