THE FUNCTION OF MEAN CONCENTRATION OF A CHANCE VARIABLE

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1. Introduction

1.1. Let X be a one-dimensional chance variable which is defined by its probability distribution function

$$Pr(X < x) = \sigma(x).$$

Thus $\sigma(x)$ is a non-decreasing function such that $\sigma(-\infty) = 0$ and $\sigma(\infty) = 1$. Let $\{X_n\}$ be a sequence of independent chance variables; that is, let the k-dimensional chance variable $(X_{i_1}, X_{i_2}, \cdots, X_{i_k})$ be defined by the condition

$$Pr(X_{i_1} < x_1, X_{i_2} < x_2, \cdots, X_{i_k} < x_k) = Pr(X_{i_1} < x_1)Pr(X_{i_2} < x_2) \cdots Pr(X_{i_k} < x_k),$$

for every finite set of distinct integers $i_1, i_2, \cdots, i_k$ and for every set of real numbers $x_1, x_2, \cdots, x_k$.

Consider the series of independent chance variables

$$\sum_{n=1}^{\infty} X_n. \tag{1.1}$$

The series (1.1) is said to converge in probability if

$$Pr(| S_n - S | > \epsilon) \to 0 \qquad \text{as } n \to \infty,$$

for every $\epsilon > 0$ for some chance variable S, where S_n denotes the partial sum $X_1 + X_2 + \cdots + X_n$. The convergence problem of (1.1) was treated by a great number of writers.

Among many results concerning the convergence problem of (1.1), there are two theories, one of which is due to A. Khintchine and A. Kolmogoroff ([11]; see also [5], [8], [12], [13] and [15], p. 142)[1] and the other due to P. Lévy ([14]; [15], pp. 130–140). A main theorem in the former theory is the one which gives the necessary and sufficient conditions for the convergence in probability of (1.1) in terms of expectations of X_i and X_i^2 under certain hypotheses. The central idea in Lévy theory is to use the function of maximum concentration.

Let the distribution function of a chance variable X be $\sigma(x)$. The function

$$Q(h) = \max_{-\infty<x<\infty} \{\sigma(x + h + 0) - \sigma(x - h - 0)\} \tag{1.2}$$

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[1] Numbers in brackets refer to the bibliography at the end.