GENERALIZED BERNOULLI AND EULER NUMBERS

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1. Several years ago Vandiver[1] introduced certain numbers which he called generalized Bernoulli numbers of the r-th order. In particular for $r = 1$, the generalized number of the first order was defined by means of

$$b_n(m, k) = (mb + k)^n, \tag{1.1}$$

where the right member is to be expanded and b^i replaced by b_i, the ordinary Bernoulli number:

$$(b + 1)^n = b^n \qquad (n > 1); \tag{1.2}$$

the m and k are arbitrary integers. In his paper Vandiver gave a theorem about $b_n(m, k)$ which reduces to the familiar Staudt-Clausen theorem when $m = 1$, $k = 0$; he has recently given another proof of this theorem.[2]

In the present note we shall first prove this theorem by means of Lucas'[3] method. In the remainder of the paper we consider certain Bernoulli polynomials in several variables and by the same method derive a theorem of the Staudt-Clausen type. For the corresponding Euler polynomials we derive congruences of Kummer's type; the method is that used by Nielsen[4] for the ordinary Euler numbers.

2. We require the well-known formula

$$b_n = \sum_{s=0}^{n} \frac{1}{s+1} \sum_{\alpha=0}^{s} (-1)^\alpha \binom{s}{\alpha} \alpha^n. \tag{2.1}$$

Expanding the right member of (1.1) and using (2.1) we get after some manipulation

$$b_n(m, k) = \sum_{s=0}^{n} \frac{1}{s+1} \Delta^s, \tag{2.2}$$

where for brevity we put

$$\Delta^s = \Delta^s(m, k) = \sum_{\alpha=0}^{s} (-1)^\alpha \binom{s}{\alpha} (m\alpha + k)^n. \tag{2.3}$$

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[1] H. S. Vandiver, *On generalizations of the numbers of Bernoulli and Euler*, Proceedings of the National Academy of Sciences, vol. 23(1937), pp. 555–559; especially p. 555.

[2] H. S. Vandiver, *Simple explicit expressions for generalized Bernoulli numbers of the first order*, this Journal, vol. 8(1941), pp. 575-584.

[3] E. Lucas, *Théorie des Nombres*, vol. 1, Paris, 1891, p. 433.

[4] N. Nielsen, *Traité Elémentaire des Nombres de Bernoulli*, Paris, 1923, p. 262.