SIMPLE EXPLICIT EXPRESSIONS FOR GENERALIZED BERNOULLI NUMBERS OF THE FIRST ORDER

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Many different explicit expressions have been given for the Bernoulli numbers, and in many ways the simplest is the following, due to Kronecker:[1]

$$b_{n-1} = \sum_{a=1}^{n} \binom{n}{a} \frac{S_{n-1}(a)}{a} (-1)^{a-1}, \tag{1}$$

where

$$S_{n-1}(a) = 0^{n-1} + 1^{n-1} + 2^{n-1} + \cdots + (a-1)^{n-1}, \qquad 0^0 = 1,$$

the b's being defined by the recursion formula $(b + 1)^n = b_n$, $n > 1$, where after expansion by the binomial theorem we set $b^k = b_k$.

In the present note we shall consider what is called by the writer the generalized Bernoulli number of the first order,[2]

$$(mb + k)^n = b_n(m, k), \tag{2}$$

where this is to be interpreted symbolically as in the expression involving b above, and where m and k are integers, $m \neq 0$. We have, obviously, $b_n = b_n(1, 0)$.

We shall derive explicit expressions for this generalized number which include (1) as a special case, and a number of more general forms for (1). It will be shown that these explicit expressions will yield a number of properties of the generalized Bernoulli numbers which include most of the known arithmetical properties of the ordinary Bernoulli numbers.

Our point of departure is the formula[3]

$$(b(m, k) + rm)^{n+1} - b_{n+1}(m, k) = m(n + 1) \sum_{i=0}^{r-1} (im + k)^n; \tag{3}$$

another proof was given by the writer.[4] Then, in particular, the special case of this when $r = 1$, which may be written

$$(b(m, k) + m)^{n+1} - b_{n+1}(m, k) = m(n + 1)k^n, \tag{4}$$

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[1] L. Kronecker, *Werke*, vol. 2, Leipzig, 1897, pp. 405–406.

[2] H. S. Vandiver, *On generalizations of the numbers of Bernoulli and Euler*, Proceedings of the National Academy of Sciences, vol. 23(1937), pp. 555–559.

[3] J. W. L. Glaisher, *On the value of certain series*, Quarterly Journal of Mathematics, vol. 31(1900), pp. 193–227; pp. 193–199.

[4] H. S. Vandiver, *An extension of the Bernoulli summation formula*, American Mathematical Monthly, vol. 36(1929), pp. 36–37.