POWER SERIES WITH MULTIPLY MONOTONIC SEQUENCES OF COEFFICIENTS

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Introduction

1. In various papers, L. Fejér[1] dealt with the following interesting theorem:

A. *Let the sequence* $\{a_n\}$ *be monotonic of order* 4. *Then the power series* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is regular and univalent for* $|z| < 1$.

Here and in what follows, a sequence $\{a_n\}$ is called monotonic of order k if all the differences

$$(1) \qquad \Delta^{(\nu)} a_n = a_n - \binom{\nu}{1} a_{n+1} + \binom{\nu}{2} a_{n+2} - \cdots + (-1)^{\nu} \binom{\nu}{\nu} a_{n+\nu}$$

are non-negative for $\nu = 0, 1, 2, \cdots, k$; $n = 0, 1, 2, \cdots$. If we write

$$(2) \qquad s_n^{(3)}(z) = \binom{n+3}{3} + \binom{n+2}{3} z + \binom{n+1}{3} z^2 + \cdots + z^n,$$

the representation

$$(3) \qquad f(z) = \sum_{n=0}^{\infty} \Delta^{(4)} a_n \cdot s_n^{(3)}(z) + (1-z)^{-1} \cdot \lim_{n \to \infty} a_n$$

holds provided the latter limit exists. This is, for instance, the case if $\Delta^{(0)} a_n \geqq 0$, $\Delta^{(1)} a_n \geqq 0$.

Obviously, the expression

$$(4) \qquad \sum_{n=0}^{\infty} A_n s_n^{(3)}(z) + A(1-z)^{-1}, \qquad A_n \geqq 0,\ A \geqq 0,$$

furnishes a parametric representation of the class of power series mentioned above.

The proof of Fejér is based on the fact that $\Re s_n^{(3)}(z)$ is decreasing when $z = re^{i\theta}$, $0 < r < 1$, and θ increases from 0 to π. This property can be extended without difficulty to non-negative linear combinations of the $s_n^{(3)}(z)$, that is,

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[1] (a) *Trigonometrische Reihen und Potenzreihen mit mehrfach monotoner Koeffizientenfolge*, Transactions of the American Mathematical Society, vol. 39(1936), pp. 18–59. (b) *Hatványsorok többszörösen monoton együtthatósorozattal*, Matematikai és Természettudományi Értesitö, vol. 55(1936), pp. 1–29. (c) *Untersuchungen über Potenzreihen mit mehrfach monotoner Koeffizientenfolge*, Acta Litterarum ac Scientiarum, vol. 8(1936), pp. 89–115.