LAPLACE TRANSFORMS OF MULTIPLY MONOTONIC FUNCTIONS

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1. **Introduction.** Let $s = \sigma + i\tau$ be a complex variable and let t be a real variable. Then if the integral in the right member of the equation

$$f(s) = \int_0^\infty e^{-st}\alpha(t)\,dt \tag{1.1}$$

converges, $f(s)$ is called the Laplace transform of $\alpha(t)$. The fundamental properties of the transformation (1.1) have been studied by Widder, Doetsch, and others (see [8], [2]). The region of convergence of the integral in the right member of (1.1) is a half-plane $\sigma > \sigma_0$ and $f(s)$ is an analytic function of s there.

In this paper we are concerned with the effect upon the structure of $f(s)$ of the assumption of various degrees of monotonic order on $\alpha(t)$. To this end we have the following

Definition of Monotonic Order. A real function $\beta(t)$ of a real variable t is said to be monotonic of order k in the interval $0 < t < \infty$ if the function satisfies the condition $(-1)^n\beta^{(n)}(t) \geqq 0$ $(n = 0, 1, 2, \cdots, k)$ in that interval.

A similar definition applies for the interval $0 \leqq t < \infty$.

Condition A. From the above definition it is clear that $\alpha(t)$ may be monotonic in character and the integral in (1.1) fail to converge as is shown by the function $\alpha(t) = 1/t^2$. Hence, throughout this paper we shall assume that, in addition to the specified monotonic character, $\alpha(t)$ and its derivatives, when these latter are assumed to exist, are such that

$$\int_0^\infty e^{-st}\alpha^{(n)}(t)\,dt$$

converges for $\sigma > 0$. Functions $\alpha(t)$ which satisfy this requirement will be said to "satisfy condition A".

Definition of $f_R(s)$. In the statement of the theorems it is convenient to use the symbol $f_R(s)$ which is defined as the integral

$$f_R(s) = \int_0^R e^{-st}\alpha(t)\,dt.$$

2. **Transforms of triply monotonic functions.**

Theorem 1. *If $\alpha(t)$ is monotonic of order three on the interval $0 < t < \infty$ and satisfies condition* A, *then*

Received April 15, 1941. A part of a thesis presented at Brown University in partial fulfillment of the requirements for the degree of Doctor of Philosophy; prepared under the direction of Professor J. D. Tamarkin.