RINGS WITH MULTIPLE-VALUED OPERATIONS

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Introduction

One of the most familiar generalizations of the concept of a "group" is that of a "multigroup" in which we retain essentially all of our postulates except that two elements combine into a subset of elements rather than a single element of the set. Since the concept of a group is fundamental in other divisions of algebra, it is natural to wonder just what one would obtain if the groups there were to be replaced by multigroups. This paper is an investigation into various parts of algebra in which this replacement has been made. It is essentially the material presented in a thesis written at the University of Illinois under the guidance of Professor Brahana. I am greatly indebted to him and to Professor Baer for many criticisms and suggestions on the material.

It shall be assumed that the reader has a general knowledge of the material on generalizations of groups.[1]

In §1 we introduce the concept of a generalized ring. This is defined much in the same way as is an ordinary ring, the fundamental difference between the two being that in our case addition and multiplication are not necessarily unique. We also obtain some elementary properties of those of our rings which are additive groups. An interesting result in this connection is that every product contains the same number n of elements, and n is a divisor of the order of the additive group.

We define an ideal which is similar to an ordinary ideal. By the use of a certain correspondence Q among the elements of our ring, we can isolate a certain subset of ideals composed of Q-ideals. We show that a satisfactory choice of the correspondence Q allows us to establish a representation of a Q-ideal which is similar to the classical one. It follows from our development, except in certain cases, that any structure considerations must be made in terms of these Q-ideals.

We consider the multiplicative properties of the elements in §4. We also discuss the case when we have unique factorization of elements into prime elements. This is done as preliminary work to §5.

In certain types of our rings we are able to define an "indeterminate domain" in which the product of two polynomials is a subset of polynomials. For some

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[1] A good bibliography may be found in M. J. Dresher and O. Ore, *Theory of multigroups*, Amer. Jour. of Math., vol. 60(1938), pp. 705–733. Other papers include J. E. Eaton and Oystein Ore, *Remarks on multigroups*, Amer. Jour. of Math., vol. 62(1940), pp. 67–71; J. E. Eaton, *Associative multiplicative systems*, Amer. Jour. of Math., vol. 62(1940), pp. 222–232; Howard Campaigne, *Partition hypergroups*, Amer. Jour. of Math., vol. 62(1940), pp. 599–612.