NON-COMMUTATIVE CHAINS AND THE POINCARÉ GROUP

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J. W. Alexander, W. Mayer, A. W. Tucker and S. Lefschetz[1] have abstracted from convex complexes an algebraic system carrying a "homology theory" which, when the algebraic system is itself a complex, becomes the ordinary homology theory of the complex. Chains are there commutative and the elements of a homology group are classes of cycles, each class being composed of cycles whose difference bounds a chain of higher dimension. It is the object of this paper to abstract from finite convex complexes in another direction to obtain an algebraic system S carrying non-commutative chains and a "homology group" π of dimension 1 whose elements are classes of cycles whose differences "bound" non-commutative chains of higher dimension. "Subdivision" of this algebraic system will be defined; the group π will be shown to be invariant under this subdivision; and, a non-trivial step in this case, it will be shown that when S is a convex complex, π is the Poincaré group.

1. The system S consists of "cells" each having associated with it an integer called its dimension and a function F (meaning boundary) whose domain is S and whose range is a subset of the "chains" of S. The cells comprise the neutral cell 1 and n-dimensional cells $\{E_i^n\}$ (called n-cells) in finite number for $n = 0, 1, 2$. It is convenient to suppose that 1 is an n-cell for each n. To simplify the notation, zero- and one-cells (or their inverses) will often be denoted by O, T, U, V and a, b, x respectively.

By an n-chain will be meant a "word" in the sense of the theory of non-Abelian groups with a finite number of generators, the letters of the word being n-cells or their inverses. For instance,

$$(1.1)\qquad \begin{aligned} C^n &\equiv (E_{i_1}^n)^{x_1} \cdots (E_{i_s}^n)^{x_s}, \qquad x_k = \pm 1, \qquad \text{and} \\ D^n &\equiv (E_{j_1}^n)^{y_1} \cdots (E_{j_t}^n)^{y_t}, \qquad y_k = \pm 1 \end{aligned}$$

are n-chains, and if

$$C^n D^n = (E_{i_1}^n)^{x_1} \cdots (E_{i_s}^n)^{x_s} (E_{j_1}^n)^{y_1} \cdots (E_{j_t}^n)^{y_t},$$

and $(E_i^n)(E_i^n)^{-1}$ are written 1, the n-chains form a free group $\mathcal{C}^n$. Obviously, more than one word defines the same element of $\mathcal{C}^n$. This distinction between the *word* C^n and the *element* C^n of $\mathcal{C}^n$ must be kept clear. A *normal form* for an element of $\mathcal{C}^n$ can be obtained from a word giving rise to that element by sup-

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[1] S. Lefschetz, Bull. Am. Math. Soc., vol. 43(1937), pp. 345–359. (References to the other authors will be found on p. 345.)