THE DOUBLE LAPLACE INTEGRAL

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The Laplace integral

$$\int_0^\infty e^{-sx} F(x)\, dx,$$

where x is real and s is real or complex, has been the subject of many extensive investigations. More recently, the Laplace-Stieltjes integral

$$\int_0^\infty e^{-sx}\, d\varphi(x)$$

has been studied (see, for example, the series of papers by D. V. Widder which appears in the Transactions of the American Mathematical Society, vols. 31, 33, 36, 39); this integral includes as special cases the ordinary Laplace integral and the Dirichlet series.

The object of this paper is to investigate the analogous integral for functions of two variables:

$$\int_0^\infty \int_0^\infty e^{-sx-ty}\, d_x d_y\, \varphi(x, y).$$

In some cases, the results are direct generalizations of the one variable theory; in others, the methods and results are quite different.

I. Introduction

1. Let $\varphi(x)$ be a complex-valued function of a real variable x, defined on the closed interval (a, b). The definitions of a function of bounded variation and of the total variation $V_\varphi[a, b]$, usually given for real-valued functions, apply equally well here. (See, for example, [18], p. 325.) If $\varphi(x) = \theta(x) + i\psi(x)$, where $\theta(x)$ and $\psi(x)$ are real, $\varphi(x)$ is of bounded variation on (a, b) if and only if $\theta(x)$ and $\psi(x)$ are of bounded variation. Hence the properties established for real-valued functions ([18], pp. 325–330) can readily be extended to complex-valued functions.

Although the concepts of positive variation $P_\varphi[a, b]$ and negative variation $N_\varphi[a, b]$ apply only to real-valued functions, the following extension of the fundamental property of functions of bounded variation is valid:

LEMMA 1. *A necessary and sufficient condition that $\varphi(x)$ be of bounded variation is that it be expressible as a linear combination, with constant complex coeffi-*

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