A GENERALIZATION OF BROUWER'S FIXED POINT THEOREM

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The purpose of the present paper is to give a generalization of Brouwer's fixed point theorem (see [1][1]), and to show that this generalized theorem implies the theorems of J. von Neumann ([2], [3]) obtained by him in connection with the theory of games and mathematical economics.

1. The fixed point theorem of Brouwer reads as follows: *if* $x \rightarrow \varphi(x)$ *is a continuous point-to-point mapping of an r-dimensional closed simplex S into itself, then there exists an* $x_0 \in S$ *such that* $x_0 = \varphi(x_0)$.

This theorem can be generalized in the following way: Let $\mathfrak{K}(S)$ be the family of all closed convex subsets of S. A point-to-set mapping $x \rightarrow \Phi(x) \in \mathfrak{K}(S)$ of S into $\mathfrak{K}(S)$ is called upper semi-continuous if $x_n \rightarrow x_0$, $y_n \in \Phi(x_n)$ and $y_n \rightarrow y_0$ imply $y_0 \in \Phi(x_0)$. It is easy to see that this condition is equivalent to saying that the graph of $\Phi(x)$: $\sum_{x \in S} x \times \Phi(x)$ is a closed subset of $S \times S$, where $\times$ denotes a Cartesian product. Then the generalized fixed point theorem may be stated as follows:

Theorem 1. *If* $x \rightarrow \Phi(x)$ *is an upper semi-continuous point-to-set mapping of an r-dimensional closed simplex S into* $\mathfrak{K}(S)$, *then there exists an* $x_0 \in S$ *such that* $x_0 \in \Phi(x_0)$.

Proof. Let $S^{(n)}$ be the n-th barycentric simplicial subdivision of S. For each vertex x^n of $S^{(n)}$ take an arbitrary point y^n from $\Phi(x^n)$. Then the mapping $x^n \rightarrow y^n$ thus defined on all vertices of $S^{(n)}$ will define, if it is extended linearly inside each simplex of $S^{(n)}$, a continuous point-to-point mapping $x \rightarrow \varphi_n(x)$ of S into itself. Consequently, by Brouwer's fixed point theorem, there exists an $x_n \in S$ such that $x_n = \varphi_n(x_n)$. If we now take a subsequence $\{x_{n_\nu}\}$ $(\nu = 1, 2, \cdots)$ of $\{x_n\}$ $(n = 1, 2, \cdots)$ which converges to a point $x_0 \in S$, then this x_0 is a required point.

In order to prove this, let Δ_n be an r-dimensional simplex of $S^{(n)}$ which contains the point x_n. (If x_n lies on the lower-dimensional simplex of $S^{(n)}$, then Δ_n is not uniquely determined. In this case, let Δ_n be any one of these simplexes.) Let $x_0^n, x_1^n, \cdots, x_r^n$ be the vertices of Δ_n. Then it is clear that the sequence $\{x_i^{n_\nu}\}$ $(\nu = 1, 2, \cdots)$ converges to x_0 for $i = 0, 1, \cdots, r$, and we have $x_n = \sum_{i=0}^{r} \lambda_i^n x_i^n$ for suitable $\{\lambda_i^n\}$ $(i = 0, 1, \cdots, r; n = 1, 2, \cdots)$ with $\lambda_i^n \geqq 0$ and $\sum_{i=0}^{r} \lambda_i^n = 1$. Let us further put $y_i^n = \varphi_n(x_i^n)$ $(i = 0, 1, \cdots, r;$

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[1] Numbers in brackets refer to the list of references at the end.