EXTENSION OF HOMEOMORPHISMS INTO EUCLIDEAN AND HILBERT PARALLELOTOPES

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Let A be a closed subset of a space X, and let f be a homeomorphism of A into a space Y. A homeomorphism f^* of X into Y is called an *extension* of f when $f^*(x) = f(x)$ for every $x \epsilon A$. If A is the null set, every homeomorphism f^* of X into Y is an extension of f. Thus the problem of extending a given homeomorphism is a generalization of the problem of imbedding. Guided by this remark I shall prove the following generalization of the Menger-Nöbeling[1] imbedding theorem:

(1) *Let A be a compact subset of a separable metrizable space X. Let n be the dimension of $X - A$, and let y be a point of the n-dimensional parallelotope*[2] E^n. *If f is a homeomorphism of A into the $(q + n)$-dimensional parallelotope $Y = E^q \times E^n$, where $q \geqq 1 + \dim X$, and if $f(A) \subset E^q \times [y]$, then f can be extended to a homeomorphism of X into Y.*

The theorem holds also in the case $n = \infty$; it is then a generalization of the Urysohn imbedding theorem[3] and reads as follows:

(2) *Let A be a compact subset of a separable metrizable space X and let y be a point of the Hilbert parallelotope*[2] E^∞. *Any homeomorphism f of A into the Hilbert parallelotope $Y = E^\infty \times E^\infty$, such that $f(A) \subset E^\infty \times [y]$, can be extended to a homeomorphism of X into Y.*

For finite q every compact subset of the Euclidean q-dimensional space R^q is contained in a homeomorph of E^q; hence E^q may be replaced by R^q in the statement of (1). In the Menger-Nöbeling theorem it is immaterial whether E^q or R^q is used, but for (1) the use of E^q results in a theorem which is *a priori* stronger.

It is interesting to compare (1) with the theorems[4] of Gehman and Adkisson-

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[1] W. Hurewicz, *Über Abbildungen von endlichdimensionalen Räumen auf Teilmengen Cartesischer Räume*, Sitzungsberichte Preuss. Akad. Wiss., vol. 24(1933), pp. 754–768, where further references may be found. Also C. Kuratowski, *Sur les théorèmes du "plongement" dans la théorie de la dimension*, Fundamenta Mathematicae, vol. 28(1937), pp. 336-342. Some of the methods of the present note stem from this latter paper.

[2] The n-dimensional parallelotope E^n is the product of the closed interval [0, 1] with itself n times. The Hilbert parallelotope E^∞ is the product of [0, 1] with itself a countable number of times. We consider E^n as a subset of Euclidean n-space R^n in the usual way, and E^∞ as a subset of Hilbert space R^∞.

[3] Alexandroff and Hopf, *Topologie*, Berlin, 1935, p. 81.

[4] H. M. Gehman, *On extending a continuous* (1–1) *correspondence of two plane continuous curves to a correspondence of their planes*, Transactions of the American Mathematical Society, vol. 28(1926), pp. 252–265; V. W. Adkisson and Saunders MacLane, *Extending maps of plane Peano continua*, this Journal, vol. 6(1940), pp. 216–228, Theorem 2, where further references may be found. Cf. also G. Choquet, *Etude des homéomorphies planes*, Paris Comptes Rendus, vol. 206(1938), pp. 159–161.