SUMS OF n-TH POWERS OF QUADRATIC INTEGERS

By Ivan Niven

1. **Introduction.** Consider the quadratic fields $R(\theta)$ defined by $\theta^2 = m$, m being a square-free rational integer not 0 or 1. Our first problem is to determine for which fields $R(\theta)$ every integer of the field is expressible as a sum of n-th powers of integers of the field. For odd powers this question is answered in §3 (Theorem 3), necessary and sufficient conditions being given in terms of m and n. For even powers, necessary and sufficient conditions are again given, but these are not as explicit as in the case of odd powers; this situation is treated in §6.

The second problem is to determine necessary and sufficient conditions that an integer of $R(\theta)$ be expressible as a sum of n-th powers of integers of the field. If n is odd, again we are able to give a complete answer; this is done in §4 (Theorems 4 and 5). For even powers we treat only imaginary fields, that is, fields for which m is negative; this is the material of §5 (Theorems 6, 7, and 8).

Quadratic integers are usually of the form $x + y\theta$, x and y being rational integers (as are all Roman letters in this paper). We shall say that such quadratic integers are of *the first kind.* When $m \equiv 1 \pmod 4$, however, quadratic integers may also be of the form $\frac{1}{2}(u + v\theta)$, u and v being odd. These will be called integers of *the second kind.* In §2 we study powers of quadratic integers of the second kind, and determine whether these powers are of the first or second kind. We show (in Theorems 1 and 2) that the n-th power of an integer of the second kind is an integer of the first kind if and only if n is divisible by 3 and m is congruent to 5 modulo 8. This result is used extensively throughout the paper, wherever integers of the second kind are under consideration.

2. **Powers of quadratic integers of the second kind.** We first prove a series of lemmas.

Lemma 1. *The product of two integers of the second kind,* $\frac{1}{2}(a + b\theta)$ *and* $\frac{1}{2}(c + d\theta)$, *is an integer of the second kind if and only if* $abcd \equiv 1 \pmod 4$.

Using the fact that m is congruent to 1 modulo 4, we see that the proof is an easy consequence of elementary congruence theory. Similarly we obtain the following result.

Lemma 2. *The product of an integer* $c + d\theta$ *of the first kind and an integer of the second kind is an integer of the second kind if and only if* c *and* d *are incongruent modulo* 2.

Lemma 3. *If an integer* $x + y\theta$ *has the property that* x *and* y *are incongruent modulo* 2, *then any power of this integer has the same property.*

Received November 20, 1940.