CHANGE OF VELOCITIES IN A CONTINUOUS ERGODIC FLOW

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Introduction. The fact that the proofs of the ergodic theorem and related theorems hold for general measure-preserving transformations has led to the formulation of the theory of flows in terms of measure theory without reference to the classical dynamical systems from which the theory arose.[1] In this paper we consider flows from this standpoint and prove some theorems about continuous ergodic flows. Our main theorem (Theorem 5) asserts that by a change of velocities any such flow can be put into a certain geometrical form; a flow in this form we call a flow built on a measure-preserving transformation. Associated with any flow is a 1-parameter group of unitary operators and the spectral resolution of such groups of operators has proved a useful tool in discussing flows.[2] Because a flow built on a measure-preserving transformation cannot have a purely continuous spectrum, this theorem can be considered as a theorem about what can be done to the spectral resolution by a change of velocities in a flow.

Roughly, our proof proceeds as follows: we show the existence of a set R (called a regular set) with the properties: (1) almost all points go in and out of R infinitely often, and (2) whenever a point gets into either R or CR, it stays there for some time interval of positive length.[3] Each of the sets R and CR is then a set of finite segments of paths of the flow. We then alter the velocities in such a way that each of these segments is traversed in the same fixed length of time. Except for a number of measurability difficulties it is easy to see that the resulting flow is built on a measure-preserving transformation.

Although we start with a continuous flow, simple examples show that we do not in general end up with a continuous flow after such a change of velocities. However, we show that the flow that we obtain will take Borel sets into Borel sets and will be a measurable flow. Since any essential change in velocities will yield a flow under which the original measure will no longer be invariant, it is necessary to find a new measure invariant under the new flow and "properly" related to the original measure. We show that there is a measure equivalent to the original measure (two measures are equivalent if they vanish for the same sets) which is invariant under the new flow.

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[1] See, for example, [3] and [6]. (Numbers in brackets refer to the bibliography at the end of this paper.)

[2] The idea of using this spectral resolution for studying flows and measure-preserving transformations is due to Koopman [4].

[3] The referee of this paper has pointed out that we have used our hypothesis that the flow be ergodic only in proving the existence of regular sets and that for this purpose it is sufficient to assume the weaker hypothesis that almost all points of the space lie on transitive (dense) paths.