A MINIMUM PROBLEM IN THE THEORY OF ANALYTIC FUNCTIONS

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Let $f(z)$ be a complex-valued function defined on $\gamma: |z| = 1$. F. Riesz[1] has investigated the problem of minimizing the integral

$$\int_{\gamma} | f(z) - p(z) | \, | dz | \tag{0.1}$$

for functions $p(z)$ which are the boundary functions on γ of functions analytic within γ. Riesz assumed $f(z)$ to be a polynomial in $1/z$. He found that there is a minimizing function $p(z) = g(z)$ (uniquely determined in the sense that two minimizing functions differ at most on a set of measure 0), which is a polynomial. The zeros of $f(z) - g(z)$ are distributed in a simple way, and Riesz' uniqueness proof depends on that fact. Riesz' problem can also be considered as the problem of minimizing $\int_{\gamma} | h(z) | | dz |$ for functions $h(z)$ which are the boundary values of functions $h(z)$ analytic within γ, a finite number of whose initial power series coefficients are given. In this form, the problem has been generalized by Kakeya,[2] who imposes other conditions on the functions $h(z)$ within γ: the values of $h(z)$ and its derivatives are prescribed at given points. In all this work, the uniqueness theorem depends essentially on the fact that the minimizing $h(z)$ can be written down explicitly, or at least that the minimizing $h(z)$ is rational with a known peculiar distribution of zeros and poles.

Before describing the problem which is to be solved in this paper, we shall state the facts which will be needed from the theory of analytic functions.[3] Let $f(z)$ be a complex-valued Lebesgue integrable function, defined on γ. Then $f(z)$ has a Fourier series:

$$f(z) \sim \sum_{-\infty}^{\infty} a_n z^n, \qquad a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z^{n+1}} \, dz = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ni\theta} \, d\theta. \tag{0.2}$$

The function $f(z)$ is said to be of power series type if $a_{-1} = a_{-2} = \cdots = 0$. If $f(z)$ is of power series type, $\sum_{0}^{\infty} a_n z^n$ converges within γ to an analytic function

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[1] F. Riesz, *Über Potenzreihen mit vorgeschriebenen Anfangsgliedern*, Acta Mathematica, vol. 42(1920), pp. 145–171.

[2] S. Kakeya, Proceedings of the Physico-Mathematical Society of Japan, (3), vol. 3(1921), pp. 48–58.

[3] See for the results summarized in this paragraph, F. Riesz, *Ueber die Randwerte einer analytischen Funktion*, Mathematische Zeitschrift, vol. 18(1923), pp. 87–95; F. and M. Riesz, *Ueber die Randwerte einer analytischen Funktion*, Compte Rendu du Quatrième Congrès des Mathématiciens Scandinaves à Stockholm (1916), pp. 27–44.