RIESZ SUMMABILITY METHODS OF ORDER r, FOR $\mathcal{R}(r) < 0$

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Given a series $\sum_{n=0}^{\infty} u_n$ of complex terms and a complex parameter r, whose real part will be denoted by $\mathcal{R}(r)$, let

$$\alpha_n = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right)^r u_k \qquad (n = 1, 2, \cdots), \tag{1}$$

and let

$$\beta(t) = \sum_{k=0}^{[t]-1} \left(1 - \frac{k}{t}\right)^r u_k \qquad (1 \leqq t < \infty).$$

By a^r, where $a > 0$, will always be meant $\exp [r \log a]$, where $\log a$ is given its real value. If $\lim_{n\to\infty} \alpha_n = L$, then $\sum u_n$ is said to be summable-A_r to L. If $\lim_{t\to\infty} \beta(t) = L$, then $\sum u_n$ is said to be summable-B_r to L. These summability methods are due to M. Riesz, and this notation is due to Agnew.[1]

If $-1 < r \leqq 1$, then A_r, B_r and the Cesàro method C_r are all equivalent,[2] while for some values of $r > 1$, A_r and B_r are not equivalent.[3] For $\mathcal{R}(r) < -1$, A_r and B_r are not equivalent.[4] For other values of r, the question of the equivalence of A_r and B_r seems not to be discussed in the literature.

The object of this note is to give a criterion for the equivalence of A_r and B_r for $\mathcal{R}(r) < 0$, based on Agnew's work, and to apply this criterion to show that A_{-1+ih} and B_{-1+ih} $(-\infty < h < \infty)$ are equivalent if and only if $h = 0$. It follows[5] that A_{-1} can take the position $r = -1$ in the scale of Cesàro summability methods C_r.

Let $\varphi_r(x) = \sum_{n=1}^{\infty} n^r x^n$, for $|x| < 1$. $\varphi_r(x)$ has a simple zero at the origin, so that we can define the coefficients $\{e_n^{(r)}\}$ by

$$f_r(x) = \frac{1}{\varphi_r(x)} - \frac{1}{x} = \sum_{n=0}^{\infty} e_n^{(r)} x^n, \qquad \text{for } |x| < R,\ R > 0. \tag{2}$$

Let $e_{-1}^{(r)} = 1$, for all r.

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[1] R. P. Agnew, *On Riesz and Cesàro methods of summability*, Transactions of the American Mathematical Society, vol. 35(1933), pp. 532–548. (See this paper for references to Riesz.)

[2] Agnew, op. cit., p. 544; and M. Riesz, *Sur l'équivalence de certaines méthodes de sommation*, Proceedings of the London Mathematical Society, (2), vol. 22(1923–24), pp. 412–419; p. 418.

[3] Agnew, loc. cit., Theorem 4.4, and Riesz, loc. cit., p. 418.

[4] Agnew, loc. cit., Theorem 10.2.

[5] Agnew, loc. cit., p. 544.