A GENERALIZATION OF THE AUMANN-CARATHÉODORY "STARRHEITSSATZ"

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1. **Introduction.** The "Starrheitssatz" of Aumann and Carathéodory [2][1] may be stated as follows:

Let G_w be a multiply-connected region of the w-plane, the boundary of which contains at least three points, and let $W = W(w)$ be analytic and single-valued for $w \in G_w$. Further let $W = W(w)$ satisfy the requirements:

(i) *there exists a $\zeta \in G_w$ such that $W(\zeta) = \zeta$,*

(ii) *$w \in G_w$ implies $W(w) \in G_w$.*

Then there exists a positive constant $\Omega(\zeta, G_w)$ less than unity such that, if $W = W(w)$ is not a (1, 1) *map of G_z onto itself, then $| W'(\zeta) | \leqq \Omega(\zeta, G_w)$.*

If we denote by C_1 the class of (1, 1) conformal maps of G_w onto itself, and by C_2 the class of all other single-valued maps which are analytic for $w \in G_w$ and have their images in G_w, then the "Starrheitssatz" asserts that there exists no sequence of maps $\{W_n(w)\}$ of C_2 with $W_n(\zeta) = \zeta$ $(n = 1, 2, \cdots)$ which converges continuously to a map of class C_1 for $w \in G_w$. Conversely, if we can establish that no map of C_1 can be expressed as the limit of a sequence of maps of class C_2, then the "Starrheitssatz" follows immediately. This results from the fact that, if $W = W_0(w)$ is a map of either class C_1 or C_2 with the properties

(i) $W_0(\zeta) = \zeta$,

(ii) $| W_0'(\zeta) | = 1$,

then $W_0(w)$ is necessarily a member of class C_1.

The "Starrheitssatz" is restrictive in its hypotheses. It requires that $\zeta \in G_w$ be a fixed point of the maps considered. It is therefore natural to seek a generalization of the "Starrheitssatz" which does not make such stringent requirements on the class of maps considered. The alleged proposition that no map of C_1 can be expressed as the limit of a sequence of maps of C_2 offers such a generalization. In this paper we shall establish a proposition of this type.

We need not restrict our attention to plane regions G_w. Instead we may very well consider abstract Riemann surfaces[2] F_w and single-valued conformal maps $W = W(w)$ of F_w into itself. We shall require that F_w be not simply-connected, that $\tilde{F}_w$, the *universal covering surface* of F_w, be of hyperbolic type. Let $w = w(z)$ denote any conformal uniformizing mapping which defines $| z | < 1$ as a smooth, unbounded covering surface of F_w. The map $w = w(z)$ is automorphic under a group $\mathfrak{G}$ of linear fractional transformations T which map $| z | < 1$ onto

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] See [6]. We adopt the notation and definitions of this text.