AN ANALOGUE OF GREEN'S THEOREM IN THE CALCULUS OF VARIATIONS

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1. **Introduction.** The purpose of the present paper is to study necessary and sufficient conditions for the equation

$$\iint_A (uz_x + vz_y + wz)\, dx\, dy = \int_C z(u\, dy - v\, dx) \tag{1}$$

to hold for every function $z(x, y)$ with continuous derivatives, where A is a region whose boundary C is composed of a finite number of rectifiable arcs without double points, every pair of which have at most an end point in common. This equation plays a fundamental rôle in the study of the properties of a minimizing surface for the double integral

$$\iint_A f(x, y, z, z_x, z_y)\, dx\, dy,$$

since the first variation of this integral is given by the first member of (1) with $u = f_{z_x}$, $v = f_{z_y}$, $w = f_z$. When w is continuous on $A + C$ and u, v have continuous derivatives on A with continuous limits on C, the criteria given below tell us that equation (1) holds if and only if $w = u_x + v_y$. Setting $z = 1$ in equation (1), one then obtains the Green's formula

$$\iint_A (u_x + v_y)\, dx\, dy = \int_C u\, dy - v\, dx.$$

On the other hand, equation (1) is an easy consequence of Green's formula when $w = u_x + v_y$. Our theorems therefore can be regarded as extensions of Green's theorem. Since our proofs are not based on Green's theorem, they can be regarded as an alternate proof of Green's theorem. The arguments here given are quite different from those given earlier by Haar,[1] Coral,[2] Bliss[3] and others.[4]

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[1] A. Haar, *Zur Variationsrechnung*, Abhandlungen aus dem mathematischen Seminar des Hamburgischen Universität, vol. 8(1930), p. 1.

[2] M. Coral, *On the necessary conditions for the minimum of a double integral*, this Journal, vol. 3(1937), pp. 585-592.

[3] G. A. Bliss, *The calculus of variations, multiple integrals*, Lectures delivered at the University of Chicago (mimeographed).

[4] See A. Huke, *An historical and critical study of the fundamental lemma of the calculus of variations, Contributions to the Calculus of Variations, 1930*, University of Chicago Press, 1931, pp. 45-160.