KHINTCHINE'S PROBLEM IN METRIC DIOPHANTINE APPROXIMATION

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It has long been known that if x is any real number there exists an infinitude of rational numbers p/q which[1] satisfy

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{1}$$

This relation raises the question: can the function $1/q^2$ on the right be replaced by a smaller function to obtain a sharper inequality? This question was answered by Hurwitz who showed that one can use the function $1/(\sqrt{5}q^2)$. Hurwitz' inequality is the "best possible" in the sense that if $1/\sqrt{5}$ is replaced by a smaller constant there are numbers x which can be approximated in the above manner only a finite number of times. An example is $x = \frac{1}{2}(1 + \sqrt{5})$.

It might be supposed, however, that by ignoring a certain limited class of numbers the inequality can be made sharper for the remaining numbers. A common method of ignoring exceptional sets is to use the idea of Lebesgue measure and to disregard sets of measure zero. The application of Lebesgue measure to improve inequality (1) was made by Khintchine[2] in 1924. However, other types of metrical problems in Diophantine approximation had been studied much earlier.

Khintchine's Theorem. *Let $\{\alpha_q\}$ be a sequence of positive numbers which satisfies*

(a) $\sum_{q=1}^{\infty} \alpha_q = \infty$,

(b) $q\alpha_q$ *is a decreasing function of* q.

Then for almost all x there exist arbitrarily many rational numbers p/q which satisfy

$$\left| x - \frac{p}{q} \right| < \frac{\alpha_q}{q}. \tag{2}$$

An example of a sequence to which this theorem applies is $\alpha_q = (q \log q)^{-1}$.

One of the results of this paper is the replacement of Khintchine's condition (b) by the weaker condition

(b′) α_q/q^c *is a decreasing function of* q.

Here c may be any real constant.

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[1] The symbols p and q will be used throughout the text to denote positive integers.

[2] A. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Mathematische Annalen, vol. 92(1924), pp. 115–125. *Zur metrischen Theorie der diophantischen Approximationen*, Mathematische Zeitschrift, vol. 24(1926), pp. 706–714.