FORMULATIONS OF THE HAUSDORFF INCLUSION PROBLEM

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1. **Introduction.** A sequence $\{x_n\}$ of complex numbers is called a *regular sequence* if there exists a *regular mass function* $\phi_x(u)$ such that[1]

$$x_n = \int_0^1 u^n \, d\phi_x(u) \qquad (n = 0, 1, 2, \cdots).$$

The conditions on $\phi_x(u)$ are that it shall be of bounded variation on the interval $0 \leqq u \leqq 1$, continuous at $u = 0$, and that

$$\phi_x(u) = \begin{cases} 0 & \text{if } u \leqq 0, \\ 1 & \text{if } u \geqq 1, \\ \frac{1}{2}[\phi_x(u-0) + \phi_x(u+0)] & \text{if } 0 \leqq u < 1. \end{cases}$$

With each regular sequence $\{x_n\}$ there is associated a *regular moment function*

$$x(z) = \int_0^1 u^z \, d\phi_x(u),$$

and a *regular moment generating function*

$$f_x(t) = x_0 - x_1 t + x_2 t^2 - \cdots = \int_0^1 \frac{d\phi_x(u)}{1 + tu}.$$

If $\{x_n\}$ is a regular sequence, the corresponding *Hausdorff transform* of a sequence $\{s_n\}$ is given by

$$t_m = \sum_{n=0}^{n=m} C_{m,n} \Delta^{m-n} x_n \cdot s_n \qquad (m = 0, 1, 2, \cdots),$$

where $C_{m,n} = m!/n!(m-n)!$, and $\Delta^i x_j = x_j - C_{i,1}x_{j+1} + C_{i,2}x_{j+2} - \cdots$. This defines a regular *Hausdorff method of summation* which is denoted by the symbol $[H, \phi_x(u)]$. Let $\{a_n\}$ and $\{b_n\}$ be two regular sequences and $[H, \phi_a(u)]$, $[H, \phi_b(u)]$ the corresponding Hausdorff methods of summation. If $b_n \neq 0$ $(n = 0, 1, 2, \cdots)$, Hausdorff showed that $[H, \phi_a(u)] \supset [H, \phi_b(u)]$ if and only if the sequence $\{a_n\}$ is *divisible* by the sequence $\{b_n\}$; i.e., $a_n = b_n c_n$ $(n = 0, 1, 2, \cdots)$,

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[1] The Stieltjes integrals discussed in this paper may always be taken in the Riemann-Stieltjes sense, but in §§3, 4 much is gained by using the Lebesgue definition instead. When integrals are to be taken in the Lebesgue-Stieltjes sense, we state so explicitly.