SIMULTANEOUS REPRESENTATION IN A QUADRATIC AND LINEAR FORM

By Gordon Pall

1. **Reduction to a single equation.** Let $c_1, \cdots, c_s, a, b$ be given integers. Consider the solvability in integers x_i of the pair of equations

$$(1) \qquad c_1x_1^2 + \cdots + c_sx_s^2 = a, \qquad c_1x_1 + \cdots + c_sx_s = b.$$

Set $u = c_1 \cdots c_s$, $t = c_1 + \cdots + c_s$, and assume $tu \neq 0$. The identity

$$(2) \qquad (\sum c_i)(\sum c_ix_i^2) - (\sum c_ix_i)^2 = \sum_{i<k}^{1,\cdots,s} c_ic_k(x_i - x_k)^2$$

suggests introducing the new variables

$$(3) \qquad y_j = x_1 - x_j \qquad (j = 2, \cdots, s),$$

whence $x_i - x_k = y_k - y_i$. Then by (1) and (2),

$$(4) \qquad ta - b^2 = \phi(y_2, \cdots, y_s),$$

where ϕ is the quadratic form, in $s - 1$ variables,

$$(5) \qquad \sum_j^{2,\cdots,s} c_j(t - c_j)y_j^2 - 2\sum_{j<k}^{2,\cdots,s} c_jc_ky_jy_k.$$

2. The author[1] treated a more general pair of equations $a = q(x_1, \cdots, x_s)$, $b = l(x_1, \cdots, x_s)$ in 1931, the coefficients of q and l being unrelated. The present article was suggested by recent work of L. E. Dickson.[2] Quite general results are obtainable by studying the form ϕ, without attempting to replace it by a form without cross-product terms. We shall consider mainly the case of positive c_i, though some of our results do not involve this restriction.

3. **Cases in which** (4) **implies** (1). If $ta - b^2$ is represented in ϕ for integers y_j, and x_i are obtained from (1_2) and (3), then $tx_1 = b + \sum c_jy_j$, and all the x_i are integers along with x_1. This proves

Theorem 1. *Let $tu \neq 0$. The number of solutions of* (1) *in integers x_i is equal to the number of solutions of* (4) *in integers y_j satisfying*

$$(6) \qquad c_2y_2 + \cdots + c_sy_s \equiv -b \pmod{t}.$$

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[1] G. Pall, Quarterly Journal of Mathematics, (Oxford), vol. 2(1931), pp. 136–143; to be referred to as QJ.

[2] L. E. Dickson, American Journal of Mathematics, vol. 56(1934), pp. 513–528. See also Dickson's *Modern Elementary Theory of Numbers*, Chicago, 1939, Chapter 10.