THE FREDHOLM THEORY OF INTEGRAL EQUATIONS

By F. Smithies

1. Introduction

1.1. Let us consider Fredholm's integral equation of the second kind:

$$x(s) = y(s) + \lambda \int k(s, t)\, x(t)\, dt, \tag{1.1.1}$$

where the integration is taken over a fixed interval (finite or infinite), and the range of the variable s is the same interval. If $k(s, t)$ is continuous, and the interval of integration is finite, the solution is given by Fredholm's famous formulas (see [3][1]):

$$x(s) = y(s) + \frac{\lambda}{d(\lambda)} \int d(s, t; \lambda)\, y(t)\, dt. \tag{1.1.2}$$

To describe the symbols appearing in (1.1.2) and in similar formulas we use the following notation. Let M_n be the determinant whose elements are $k(u_i, u_j)$ $(i, j = 1, 2, \cdots, n)$. Let N_n be the determinant obtained by replacing the elements on the main diagonal of M_n by zero. Let $M_n^*(s, t)$ be the determinant obtained by bordering M_n thus

$$\left| \begin{array}{c|ccc} k(s, t) & k(s, u_1) & \cdots & k(s, u_n) \\ \hline k(u_1, t) & & & \\ \cdots & & M_n & \\ k(u_n, t) & & & \end{array} \right|$$

and let N_n^* be obtained similarly from N_n. In this notation we have

$$d(\lambda) = 1 - \lambda \int k(u, u)\, du + \frac{\lambda^2}{2!} \iint M_2\, du_1\, du_2 - \cdots,$$

$$d(s, t; \lambda) = k(s, t) - \lambda \int M_1^*(s, t)\, du_1 + \frac{\lambda^2}{2!} \iint M_2^*(s, t)\, du_1\, du_2 - \cdots,$$

provided that λ is not a zero of $d(\lambda)$. The series are convergent for all finite complex values of λ.

In the present paper I wish to discuss the solution of the equation (1.1.1) when all we know about $k(s, t)$ is that it is a measurable function of (s, t) and that

$$\iint | k(s, t) |^2 ds\, dt < \infty.$$

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[1] Numbers in square brackets refer to the bibliography at the end of the paper.