THE GROWTH OF THE SOLUTIONS OF A DIFFERENTIAL EQUATION

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1. It has been shown[1] that the solutions of the differential equation

$$\frac{d^2x}{dt^2} + \phi(t)x = 0 \tag{1.1}$$

are bounded as $t \to \infty$ if there exists a constant $a > 0$ such that

$$\int_0^\infty | \phi(t) - a | \, dt < \infty. \tag{1.2}$$

It has also been shown[2] that if

$$\phi(t) = a + O\left(\frac{1}{t}\right), \tag{1.3}$$

the solutions need not be bounded. Here we shall go further in this direction and show that (1.2) is actually a best possible condition. We shall also show that if (1.2) is satisfied the solutions of (1.1) are not only bounded, but also resemble the solutions of the differential equation $x'' + ax = 0$ in another sense. Namely, if (1.2) is satisfied, then any solution of (1.1) satisfies also

$$\limsup_{t \to \infty} | x(t) | > 0.$$

In fact what we shall show is that the rapidity with which the solutions of (1.1) can grow and the rapidity with which they can tend to zero both depend on the growth of $\alpha(t)$, where

$$\alpha(t) = \int_0^t | \phi(t) - a | \, dt. \tag{1.4}$$

Thus the results for (1.2) will be a particular case of (1.4) where $\alpha(t)$ is bounded, and in this sense the first result we shall prove, Theorem I, is a generalization of the result of Fukuhara and Nagumo.

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[1] Fukuhara and Nagumo, *On a condition of stability for a differential equation*, Proc. Imp. Acad. of Japan, vol. 6(1930), pp. 131–132.

[2] O. Perron, *Über ein vermeintliches Stabilitätskriterium*, Göttinger Nachrichten, Math. Phys. Klasse, 1930, pp. 28–29, equation (6).