SYMMETRIC TRANSFORMATIONS IN HILBERT SPACE

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At various points in the further development of the abstract theory of boundary conditions [1],[1] and in the applications of this theory to differential equations, certain elementary questions concerning symmetric transformations[2] in Hilbert space have arisen which have escaped formal notice in the previous literature. It is the purpose of this note to dispose of these matters.

The questions with which we are concerned refer to a closed linear symmetric transformation H with the property that $(H - \lambda I)^{-1}$ exists and is bounded for some real λ. Such a symmetric transformation always possesses a self-adjoint extension, and here we show that the deficiency-index of H is (n, n), where n is the dimension number of the manifold of solutions of the equation $H^*f - \lambda f = 0$. In addition, we show that under the condition stated, H has a self-adjoint extension S with λ in its resolvent set. Finally, assuming one such extension to be known, we obtain a characterization of all maximal symmetric extensions of H in terms of maximal symmetric transformations in the manifold of zeros of $H^* - \lambda I$ and its subspaces.

We use throughout the paper the notation and terminology of the treatise of Stone [6], except for minor modifications; in particular, we denote the domain, range, and graph of a transformation T in Hilbert space $\mathfrak{H}$ by $\mathfrak{D}(T)$, $\mathfrak{R}(T)$, and $\mathfrak{B}(T)$, respectively.

Before proceeding, we note a simple theorem which permits us, without affecting the generality of the results, to reduce our problem to the case that the number λ appearing in the condition described above is zero.

Theorem 1. *Let H be a closed linear symmetric transformation in Hilbert space $\mathfrak{H}$. Then S is a symmetric extension of H if and only if $S - \lambda I$ is a symmetric extension of $H - \lambda I$ ($\mathfrak{I}(\lambda) = 0$). Moreover S and $S - \lambda I$ have the same deficiency-index.*

The first assertion is obvious, and the second is an immediate consequence of elementary facts in the general theory of symmetric transformations.[3]

We consider then a closed linear symmetric transformation H such that H^{-1} is bounded and proceed to construct a self-adjoint extension S of H, also with bounded inverse. In order to effect this construction we first note that since H^{-1} is bounded, $\mathfrak{R}(H)$ is closed. Thus H^{-1} can be regarded as a linear trans-

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[1] Numbers in brackets designate the references listed at the end of the paper.

[2] For the general theory of symmetric transformations see [4], especially pp. 80–91, or [6], Chapter 9.

[3] [6], Theorem 9.8.