FUNCTIONS WITH POSITIVE DIFFERENCES

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This note had its origin in an effort to make accessible in the literature the proof of a lemma used by Widder[1] in the study of the bilateral Laplace transform. The result in question was that a continuous function which has all differences of even order non-negative in an interval is necessarily analytic there. That it was true was fairly evident from earlier work of S. Bernstein,[2] though the details required for its demonstration seemed not to be available. Following a very natural inductive method we were easily able to supply a proof. But we soon saw that our method would prove a great deal more. We were able to show in fact that a continuous function which has a single difference, say of order k, of constant sign throughout an interval is of class C^{k-2} there, has right-hand and left-hand derivatives of order $k - 1$, and is convex or concave in pieces as if it had a k-th derivative of constant sign. This proved, it is clear that the function of the lemma has derivatives of all orders. A theorem of S. Bernstein[2] then guarantees its analyticity. But a new proof of this and related facts will be given by Boas in a separate note.

After completing our proof we discovered that the result had been proved earlier by T. Popoviciu.[3] Since our method is simpler and more direct for the purpose in hand, we believe its publication will be of value. We point out one main difference in the two methods of attack. Popoviciu makes his discussion depend on divided differences involving unequally spaced points, whereas we deal entirely with differences involving only equally spaced points.

DEFINITION 1.

$$\begin{aligned} \Delta_\delta^0 f(x) &= f(x), \\ \Delta_\delta^1 f(x) &= \Delta_\delta f(x) = f(x+\delta) - f(x), \\ \Delta_\delta^k f(x) &= \Delta_\delta^{k-1} f(x+\delta) - \Delta_\delta^{k-1} f(x) \qquad (k = 2, 3, \cdots). \end{aligned}$$

It is easy to establish by induction the useful formula

$$\Delta_\delta^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + i\delta).$$

Received September 21, 1940; presented to the American Mathematical Society, September 10, 1940.

[1] D. V. Widder, *Necessary and sufficient conditions for the representation of a function by a doubly infinite Laplace integral*, Bulletin of the American Mathematical Society, vol. 40(1934), pp. 321–326.

[2] S. Bernstein, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle*, Paris, 1926, pp. 190–197.

[3] T. Popoviciu, *Sur l'approximation des fonctions convexes d'ordre supérieur*, Mathematica (Cluj), vol. 8(1934), pp. 1–85, especially pp. 54–58.