CONDITIONS ON THE NODES OF A RATIONAL PLANE CURVE

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1. Introduction. The rational curve in the plane of order $n + 3$ with rational parameter t, say $\rho_2^{n+3}(t)$, has a set of $N_n = \binom{n+2}{2}$ nodes at points p_h, say the set $P^2_{N_n}$. If $n < 3$, these nodes may be taken generically and the rational curve exists. For $n = 3$, however, the ten nodes are subject to three conditions first noticed by Valentiner in 1881.[1] For $n > 3$ similar conditions have not been obtained. It is the purpose of this paper to obtain necessary conditions on the set $P^2_{N_n}$ for generic n. For this we define in §4 a set of N_n "nodular" points $P'^2_{N_n}$, which is shown to be not projective to $P^2_{N_n}$ in general. In §5 a set of N_n "catalectic" points $Q^n_{N_n}$ appears in the space $[n]$ of the conjugate rational envelope $r_n^{n+3}(t)$. We prove in §7 that there is a trilinear form $T(2, n, n + 1) = (\alpha x)(\beta y)(\gamma z)$ with pairs $x, y = p_h, q_h$ drawn from $P^2_{N_n}, Q^n_{N_n}$ which are neutral for z in $T = 0$; and in §6 that there is a trilinear form $T'(2, n, n + 1)$ with neutral pairs $x', y = p'_h, q_h$ drawn from $P'^2_{N_n}, Q^n_{N_n}$ which are neutral for z in $T' = 0$. The identity of the set $Q^n_{N_n}$ for the non-projective sets $P^2_{N_n}, P'^2_{N_n}$ yields the conditions desired. These conditions are sufficient for $n = 3$. We obtain the forms T, T' from the rational curve and its conjugate rational envelope by using certain special coördinate systems developed in §§2, 3. The pertinent theory of such forms is given in a recent paper.[2]

2. A special coördinate system in $[n + 1]$. Two binary forms of order $n + 1$,

$$(1)\qquad \begin{aligned} (\alpha t)^{n+1} &= (\alpha_0 t_0 + \alpha_1 t_1)^{n+1} = \sum_j \binom{n+1}{j} a_j t_0^{n+1-j} t_1^j, \\ (\beta t)^{n+1} &= (\beta_0 t_0 + \beta_1 t_1)^{n+1} = \sum_j \binom{n+1}{j} b_j t_0^{n+1-j} t_1^j \quad (j = 0, \cdots, n+1), \end{aligned}$$

have a bilinear invariant

$$(2)\qquad (\alpha\beta)^{n+1} = \sum_j (-1)^j \binom{n+1}{j} a_j b_{n+1-j}.$$

If then we put these forms into correspondence with respectively the primes ζ and the points z of a space $[n + 1]$ by setting

$$(3)\qquad \zeta_j = a_j, \qquad z_j = (-1)^j \binom{n+1}{j} b_{n+1-j},$$

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[1] For references cf. A. B. Coble, *The ten nodes of the rational sextic and of the Cayley symmetroid*, Amer. Jour. of Math., vol. 41(1919), pp. 243-265; pp. 251-254.

[2] A. B. Coble, *Trilinear forms*, this Journal, vol. 7(1940), pp. 380-395.