THE LATTICE POINTS OF AN n-DIMENSIONAL TETRAHEDRON

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In this paper we consider the problem of determining the number of lattice points inside or on the boundary of the n-dimensional simplex or "tetrahedron" bounded by the n coördinate hyperplanes

$$x_1 = 0, x_2 = 0, \cdots, x_n = 0$$

and the hyperplane

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n = \lambda,$$

where ω_i are positive and λ is a non-negative parameter. Points on the boundary are given the same weight as interior points. The total number of such points we denote by

$$N_n = N_n(\lambda) = N_n(\lambda \mid \omega_1, \omega_2, \cdots, \omega_n).$$

In other words, N_n is the number of sets $(x_1, x_2, \cdots, x_n)$ of non-negative integers for which the inequality

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n \leqq \lambda \tag{1}$$

holds. Although the right triangle case ($n = 2$) has been considered by many writers,[1] there is as yet no published account of the general problem for $n > 2$. There are a number of isolated problems, however, which have been treated from time to time and which may be considered as special cases of the higher dimensional tetrahedron. It is the purpose of this paper to present a workable method for obtaining inequalities for the function $N_n(\lambda)$.

Three special tetrahedra may be mentioned as outstanding examples: (1) the equilateral tetrahedron, (2) the "additive" tetrahedron, in which the ω's are distinct integers, and (3) the "multiplicative" tetrahedron in which the ω's are logarithms of primes.

The first of these cases is the only one in which a really simple formula for $N_n(\lambda)$ can be given, and is useful for comparing approximate formulas. This case is interesting also as being that in which $N_n(\lambda)$ has the greatest discontinuity. The other two cases, which are interesting on account of their applications, will be considered briefly in what follows.

Before considering any special tetrahedra, however, we set down a fundamental recursion formula for the general tetrahedron obtained by dissecting the tetrahedron by the parallel hyperplanes

$$x_n = k \qquad (k = 0, 1, 2, \cdots).$$

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[1] See Koksma, *Diophantische Approximationen*, Berlin, 1936, pp. 102–110.