THE DISSECTION OF RECTANGLES INTO SQUARES

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Introduction. We consider the problem of dividing a rectangle into a finite number of non-overlapping squares, no two of which are equal. A dissection of a rectangle R into a finite number n of non-overlapping squares is called a *squaring* of R of *order* n; and the n squares are the *elements* of the dissection. The term "elements" is also used for the lengths of the sides of the elements. If there is more than one element and the elements are all unequal, the squaring is called *perfect*, and R is a *perfect rectangle*. (We use R to denote both a rectangle and a particular squaring of it.) Examples of perfect rectangles have been published in the literature.[1]

Our main results are:

Every squared rectangle has commensurable sides and elements.[2] (This is (2.14) below.)

Conversely, every rectangle with commensurable sides is perfectible in an infinity of essentially different ways. (This is (9.45) below.) (**Added in proof.** Another proof of this theorem has since been published by R. Sprague: Journal für Mathematik, vol. 182(1940), pp. 60–64; Mathematische Zeitschrift, vol. 46(1940), pp. 460–471.)

In particular, we give in §8.3 a perfect dissection of a square into 26 elements.[3]

There are no perfect rectangles of order less than 9, and exactly two of order 9.[4] (This is (5.23) below.)

The first theorem mentioned is due to Dehn, who remarked[5] that the difficulty of the problem is the semi-topological one of characterizing how the elements fit together. This is overcome here in §1 by associating a certain linear graph (the "normal polar net") with each "oriented" squared rectangle. The metrical properties of the squared rectangle are found to be determined by a certain flow of electric current through this network. Accordingly, in §2 we collect the relevant results from the theory of electrical networks. In particular, the elements of the squared rectangle can be calculated from determinants formed from the incidence matrix of the network. In §3, the elements are expressed in a different way, in terms of the subtrees of the network. This leads

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[1] A bibliography is given at the end of this paper. Numbers in square brackets refer to this bibliography.

[2] Cf. [6], p. 319.

[3] This disproves a conjecture of Lusin; cf. [10], p. 272. For an independent example of a perfect square (published while this paper was in preparation) see [13].

[4] Partly confirming and partly disproving a conjecture of Toepken (see [18]).

[5] [12], p. 402.