THE FUNDAMENTAL SOLUTION OF THE PARABOLIC EQUATION

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1. **Introduction.** The function

$$\frac{1}{[4\pi(y-\eta)]^{\frac{1}{2}n}} \exp\left(\frac{-\sum_{i=1}^{n}(x_i-\xi_i)^2}{4(y-\eta)}\right) \qquad (y>\eta) \tag{1}$$

is known as the fundamental solution of the parabolic equation

$$\Delta u - \frac{\partial u}{\partial y} = 0 \qquad \left(\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}\right).$$

Following a method of successive approximations introduced by Hadamard[1] for the case $n = 1$, Gevrey,[2] using the function (1) as the first approximation, showed the existence of a fundamental solution of the equation

$$\Delta u + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + au - \frac{\partial u}{\partial y} = 0. \tag{2}$$

If in equation (2) we replace Δu by an elliptic operator

$$H(u) = \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right),$$

then the function in (1) is no longer available as the first approximation of the fundamental solution of this new equation. For $n < 3$, this new equation can be transformed into the equation (2), but for $n > 2$, this is not the case. Thus for $n > 2$, the existence of a fundamental solution is not shown by Gevrey's method.

In case the a_{ij} in $H(u)$ are not functions of the variable y, Rothe[3] has shown that the equation

$$H(u) - \frac{\partial u}{\partial y} = 0$$

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[1] J. Hadamard, *Sur la solution fondamentale des équations aux dérivées partielles du type parabolique*, Paris Comptes Rendus, vol. 152(1911), pp. 1148-1149. For another treatment of this case see W. Feller, *Zur Theorie der stochastischen Prozesse*, Math. Annalen, vol. 113(1936-37), pp. 113-160.

[2] M. Gevrey, *Sur les équations aux dérivées partielles du type parabolique*, Journal de Mathématiques, (6), vol. 10(1913), pp. 105-148.

[3] E. Rothe, *Über die Grundlösung bei parabolischen Gleichungen*, Math. Zeitschrift, vol. 33(1931), pp. 488-504.