UNIVALENT DERIVATIVES OF ENTIRE FUNCTIONS

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Let $f(z)$ be an entire function, and let $M(r)$ denote the maximum of $|f(z)|$ in $|z| \leqq r$. The object of this note is to establish the existence of a positive number T for which the following theorem is true.

Theorem. *If the entire function $f(z)$ satisfies*

$$\limsup_{r\to\infty} \frac{1}{r} \log M(r) < T, \tag{1}$$

and $f(z)$ is not a polynomial, an infinite number of the derivatives of $f(z)$ are univalent in the unit circle, $|z| \leqq 1$.

It will be shown that a possible value for T is log 2; I do not know whether or not this is the best possible value.

The following corollary is immediately obtainable by a change of variable and an application of the diagonal process.

Corollary. *If $f(z)$ is an entire function, not a polynomial, of order less than one, or of order one and minimum type, then corresponding to any increasing sequence of numbers r_n there is an increasing sequence of integers k_n such that $f^{(k_n)}(z)$ is univalent in $|z| < r_n$ $(n = 1, 2, \cdots)$.*

I show first that if $f(z)$ satisfies (1), with sufficiently small T, and neither $f(z)$ nor any derivative is univalent in the unit circle, then $f(z)$ is a constant. If neither $f(z)$ nor any derivative is univalent, there exist numbers a_n, b_n, such that for $n = 1, 2, \cdots$,

$$\begin{gathered} |a_n| \leqq 1, \; |b_n| \leqq 1, \; a_n \neq b_n, \\ f^{(n-1)}(a_n) = f^{(n-1)}(b_n). \end{gathered} \tag{2}$$

Without loss of generality, we may assume

$$f(0) = 0. \tag{3}$$

Consider the functions $h_n(z)$ defined as follows:

$$h_0(0) = 0,$$

$$z^n[1 + h_n(z)] = z^{n-1} \frac{e^{a_n z} - e^{b_n z}}{a_n - b_n} \qquad (n = 1, 2, \cdots).$$

It is obvious that $h_n(0) = 0$, since

$$1 + h_n(z) \to 1 \qquad (z \to 0).$$

Received February 23, 1940.