INTEGRATION IN ABSTRACT SPACE

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1. **Introduction.** The problem of integration in abstract space has been approached from many angles.[1] The most interesting, from the standpoint of simplicity and usefulness, is that of Bochner. He considers functions $f(x)$ on a range of measurable sets to a complete normed vector space X. The function $f(x)$ is measurable if it is the limit of a sequence of finite-valued functions, and is integrable if it is measurable and if the real function $\| f(x) \|$ is summable. There are some simple functions to which Bochner's theory does not assign an integral, one of which is:

Let X be the space of bounded functions $x(t)$ on $0 \leqq t \leqq 1$. Let $f(x) = x(t)$, where $x(t) = 0$, $0 \leqq t < x$, $x(t) = 1$, $x \leqq t \leqq 1$, and let $\| f(x) \|$ be the least upper bound of $| x(t) |$, $0 \leqq t \leqq 1$.

The difficulty here is that $f(x)$ is not the limit of a sequence of finite-valued functions. The theory set forth by Birkhoff has all the generality that can reasonably be hoped for. It is, however, somewhat removed from the simplicity that characterizes the work of Bochner and the classical theories for real and complex variables. The present paper started in an attempt to formulate Bochner's definition of a measurable function in terms of the behavior of the function apart from its relation to any sequence. The outcome is a theory of integration which is equivalent to that of Birkhoff, and consequently includes that of Bochner, while the developments are more in the spirit of the classical theories for real and complex variables.

2. **Definition of integrability.** Let X be a complete normed vector space. Let E be any set of elements x on which a measure function has been defined. For the sake of definiteness we take E to be a bounded Lebesgue measurable set in Euclidian space, and x the points of E. Let $f(x)$ be a function defined on E to X. We first define integrability for bounded functions.

Let $f(x)$ be bounded on E. If there exists a sequence of measurable sets

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