RESIDUATED DISTRIBUTIVE LATTICES

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I. Introduction

1. Although the lattice analogues of E. Noether's decomposition theorems for the ideals of commutative rings are independent of the modular or distributive condition,[1] it is of interest to consider the decompositions when such conditions are imposed. We study here the frequently occurring case when the basic lattice is distributive.[2]

Let $\mathfrak{S}$ be a residuated lattice in which the ascending chain axiom holds. If every irreducible of $\mathfrak{S}$ is primary, $\mathfrak{S}$ is called a "Noether lattice"; for such lattices, all the decomposition theorems of general commutative ideal theory are valid. For brevity, call such a lattice an N-lattice.

If $a \supset b$ in $\mathfrak{S}$ if and only if there exists an element c such that $ac = b$, $\mathfrak{S}$ is called a "principal element lattice", or for short a P-lattice. A P-lattice is distributive (W-D [1], Theorem 13.2) and so is every lattice in which any one of the following three equivalent conditions holds (W-D [1], Theorem 13.1):

$$(1.1) \qquad a:b \cup b:a = i; \qquad a:(b \cap c) = a:b \cup a:c; \qquad (b \cup c):a = b:a \cup c:a.$$

Here i is the identity element of the lattice.

A P-lattice is necessarily an N-lattice (W-D [1], Theorem 13.3). There exist, however, finite residuated lattices in which conditions (1.1) hold but which are not N-lattices. A Noether lattice in which conditions (1.1) hold will be called a "semi-arithmetical lattice", or SA-lattice.

2. Our main results are as follows. Necessary and sufficient conditions that an N-lattice be a P-lattice are, first, that an element be irreducible if and only if it is a power of a prime; and secondly, if p is a prime and q any proper divisor of p, then $qp = p$. A necessary and sufficient condition that an N-lattice be an SA-lattice is that any two primaries of the lattice be either coprime or else divisible one by the other.

In both types of lattice, the Noether decomposition of an element as a cross-cut of primary components is unique; this unicity is not a property of an arbitrary distributive Noether lattice as we show by an example, contrary to a statement in W-D [1].

In both an SA-lattice and a P-lattice, every element has a unique reduced

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[1] See the paper Ward-Dilworth [1] cited at the close of this article. We shall refer to this paper as W-D [1]. Numbers in brackets refer to the list of references at the end.

[2] The previous investigations in Ward [1], [2] appear here as special cases.