AN AXIOMATIC CHARACTERIZATION OF L_p-SPACES

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1. **Introduction.** The spaces

(a) $l_{p,n}$, $l_{p,\infty}$ $(1 \leqq p < \infty)$, whose elements are finite or infinite sequences $x = \{\xi_n\}$ with a finite norm $\| x \| = (\sum | \xi_n |^p)^{1/p}$,

(b) L_p $(1 \leqq p < \infty)$, whose elements are Lebesgue measurable functions over the interval (0, 1) with a finite norm $(\int | f(t) |^p \, dt)^{1/p}$,

(c) any direct sum of (a) and (b) with the same number p, where the norm is given as the $l_{p,2}$ norm of the norms of the components,

(d) $l_{\infty,n}$, whose elements are $\{\xi_1, \cdots, \xi_n\}$ with the norm max $(| \xi_n |)$,

(e) c_0, whose elements are sequences converging to zero with the norm max $(| \xi_n |)$,

are examples of separable, normed, complete, linear spaces (i.e., separable Banach spaces). They can be partially ordered by defining $x_1 < x_2$, if $\xi_n^{(1)} \leqq \xi_n^{(2)}$, for all n, when the space is a sequence space, $f_1(t) \leqq f_2(t)$ a.e. when the space is a function space. This partial ordering satisfies all axioms of §2. Finally, all these spaces have the following property:

PROPERTY P. *If* $a = a_1 + a_2$, *where* a_1 *and* a_2 *are orthogonal, if* $b = b_1 + b_2$, *where* b_1 *and* b_2 *are orthogonal, and if*

$$\| a_1 \| = \| b_1 \|, \qquad \| a_2 \| = \| b_2 \|,$$

then $\| a \| = \| b \|$.

The purpose of the present paper is to show that this property is characteristic for the spaces considered. Precisely, we prove the

THEOREM. *Any partially ordered, separable, Banach space of at least three dimensions in which property* P *is valid is strongly equivalent* (cf. §7 for terminology) *to one of the above-mentioned spaces.*

Some results are also obtained when the separability is not assumed.

2. Partially ordered spaces.[1]

2.1. **Axioms.** A linear space is said to be a partially ordered linear space if it satisfies the following axioms:

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[1] Partially ordered spaces were considered first by F. Riesz (Proceedings of the International Congress of Mathematicians, Bologna, vol. 3, 1928, pp. 143–148) and by L. Kantorowitch (cf., in particular, *Lineare halb-geordnete Räume,* Recueil Mathématique, new series, vol. 2(1937), pp. 121–165). The present paper is based on H. Freudenthal's paper *Über teilweise geordnete Moduln,* Proceedings, Amsterdam Academy, vol. 39(1936), pp. 641–651. This paper will be cited as F. The axioms which we are postulating correspond to those in this paper up to and including (6.3).