LAGUERRE POLYNOMIALS AND THE LAPLACE TRANSFORM

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Introduction. The object of this paper is to study the Laplace transform

$$(1) \qquad F(t) = \int_0^\infty e^{-tu}\phi(u)\,du, \qquad R(t) > t_0 > 0,$$

where

$$\int_0^\infty e^{-u} |\phi(u)|^2\,du$$

exists, by means of the orthonormal Laguerre polynomials

$$\phi_n(u) = \phi_n(u; 0, \infty; e^{-u}) = \frac{1}{n!}\left(u^n - \frac{n^2}{1!}u^{n-1} + \frac{n^2(n-1)^2}{2!}u^{n-2} + \cdots\right),$$

$$(2) \qquad \int_0^\infty e^{-u}\phi_n(u)\phi_m(u)\,du = \delta_{m,n} \qquad (m, n = 0, 1, 2, \cdots).$$

The connection between the Laplace integral and Laguerre polynomials has been exhibited by Widder [15][1] who made use of it in order to obtain an inversion formula for the general Laplace transform

$$(3) \qquad \int_0^\infty e^{-tu}\,d\alpha(u) \qquad (\alpha(u) \text{ bounded, non-decreasing}).$$

Here we are concerned mainly with the uniqueness of the representation (1) and with the nature of $F(t)$. Our discussion is based upon the now classical property of Laguerre polynomials expressed in Parseval's formula

$$\int_0^\infty e^{-u}f_1(u)f_2(u)\,du = \sum_{n=0}^\infty A_nB_n,$$

where

$$A_n = \int_0^\infty e^{-u}f_1(u)\phi_n(u)\,du, \qquad B_n = \int_0^\infty e^{-u}f_2(u)\phi_n(u)\,du.$$

(4), where the series converges absolutely, holds for any two functions $f_{1,2}(u)$ such that $\int_0^\infty e^{-u} |f_{1,2}(u)|^2\,du$ exists[2] (which implies the existence of

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[1] Numbers in brackets refer to the bibliography at the end.

[2] $\phi(u)$ in (1) and also $f(u), f_1(u), \cdots$ are in general complex-valued functions of the real variable u; e.g., $f_1(u) = \psi_1(u) + i\psi_2(u)$, where the ψ's are real.