THE RING OF AUTOMORPHISMS OF AN ABELIAN GROUP

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Let A be an Abelian group (written additively) all of whose elements have a finite order. For the purposes of this study, it suffices to suppose that A is a primary group, of characteristic p. We shall consider only a restricted class of primary groups, namely, those for which every element different from 0 has a finite height.[1]

A first step in the study of the automorphisms is to describe those subgroups N which are mapped into themselves by all the (proper and improper) automorphisms α of A, $N\alpha \subset N$. These will be called *normal* subgroups.[2] The construction of all the normal subgroups of A is contained in Part I of this paper.[3] Every normal subgroup turns out to be generated by, and to be the intersection of, irreducible normal subgroups (Theorems 4, 5).

Let $\mathfrak{o}$ be the ring of all the automorphisms of A. In Part II, we establish a one-to-one correspondence between the normal subgroups of A and certain two-sided ideals of $\mathfrak{o}$, the *normal* ideals. A right normal ideal $\mathfrak{r}$ is a largest ideal which annihilates a given ideal $\mathfrak{a}$ on the right, i.e., for which $\mathfrak{a} \cdot \mathfrak{r} = 0$. To every normal subgroup N of A corresponds the totality of automorphisms in $\mathfrak{o}$ which map N into 0. This is shown to be a right normal ideal, and an inverse correspondence is established. Similarly for left normal ideals. Theorems 6, 7, 8 and 9 form the main results.

The correspondence established between normal subgroups of A and normal ideals of $\mathfrak{o}$ permits carrying over to normal ideals theorems on normal subgroups (Theorems 10–14). The most noteworthy of these results are:

the join and intersection of normal ideals are normal ideals;

the two distributive laws hold for normal ideals; and

every normal ideal is the join and also intersection of irreducible normal ideals.

A similar theory might be developed for the group $\mathfrak{G}$ of proper automorphisms of A, but it suffers from various defects. It is pointed out in §8 that such a

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[1] The height of an element a is the largest number m for which the equation $a = p^m x$ has a solution x in A.

[2] The names "regular characteristic", "$\mathfrak{o}$-characteristic", etc. have been used for these subgroups.

[3] Characteristic subgroups have been described by Miller, Shoda, and Baer. Cf. G. A. Miller, *Determination of all the characteristic subgroups of an Abelian group*, Quart. Journ. of Math., vol. 50(1923), pp. 54–62; K. Shoda, *Über die charakteristischen Untergruppen einer endlichen Abelschen Gruppe*, Math. Zeitsch., vol. 31(1930), pp. 611–624; R. Baer, *Types of elements and the characteristic subgroups of Abelian groups*, Proc. London Math. Soc., (2), vol. 39(1935), pp. 481–514. Our procedure is closely akin to that of Baer.