THE APPROXIMATION OF FUNCTIONS SATISFYING A LINEAR PARTIAL DIFFERENTIAL EQUATION

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1. **Statement of the problem.** The determination of the solution of a linear differential equation satisfying given boundary data is a classical problem in the theory of these equations. For instance, in the case of elliptic equations[1] values on the boundary of a domain may be given, and it is necessary to search for the function which satisfies the equation inside the domain and which assumes the given values on the boundary (first boundary problem). If, moreover, these values are equal to zero, it is necessary in certain cases to determine a parameter in the equation for which there exists a non-identically vanishing solution which is zero on the boundary (characteristic value problem). In the case of hyperbolic equations, values of the function and of its first derivatives are known on the initial line, which is cut by every characteristic line at one point only; from these data it is necessary to determine the solution in a certain domain (Cauchy's problem). In addition to the problem of the proof of the existence of solutions of this kind, there is the problem of finding a method of calculating numerically the desired solution from the given data.[2]

In the papers [2, 5, 6, 7][3] stress has been laid on a connection between functions $F(x, y)$ satisfying a linear partial differential equation in two real variables x, y and certain classes of functions $f(z)$ of one complex or real variable. In fact, given an equation $\mathbf{L}(U) = 0$, it is possible to find a linear operation which transforms an arbitrary function $f(z)$ of a certain class into a particular solution of $\mathbf{L}(U) = 0$. As a well-known special case, we obtain a harmonic function (i.e., a function satisfying Laplace's differential equation) if we take the real part of any analytic function $f(z)$.

This relation enables us to transfer certain theorems of the theory of functions of one variable to functions which satisfy a linear differential equation. In particular, we approximate arbitrary solutions of a differential equation by linear combinations of a system of particular solutions of the *same* equation; and in this manner deal with the problems mentioned above.

The following two problems arise (both of which will be solved in the present paper):

(1) Given a differential equation $\mathbf{L}$, to give a procedure for calculating a

Received March 17, 1939.

[1] Elliptic (hyperbolic) equation means linear differential equation of elliptic (hyperbolic) type.

[2] The proof of the existence of a solution often does not enable one to calculate this solution; e.g., if we use the method of choice. Often, also, the determination of the solution along the lines indicated in the proof leads to enormous numerical calculations.

[3] Numbers in brackets refer to the bibliography.